Linear systems and examples of surfaces of general type

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Let $X$ be an algebraic surface and $k(X)$ its field of rational functions. Given $x \in X$, define the local ring $O_{X,x} := \{ f \in k(X) | f \text{ regular at } x \}$. Given $U \subset X$, let $O(U) = \bigcap_{x \in U} O_{X,x}$ be the structure sheaf, i.e. the sheaf of regular functions on $U$. An irreducible curve $C \subset X$ is called a prime divisor. In this case, if $U \subset X$ is affine, then $I(C \cap U) \subset O(U)$ and $P = I(C \cap U)$ is a prime ideal.

1 Divisors

Basic Fact. If $x \in X^\text{sm}$, then $O_{X,x}$ is a UFD. In particular, if $C$ is a prime divisor containing $x$, then $I(C) \cap O_{X,x} = (\pi)$. There exists an affine chart $x \in U$ such that $I(C \cap U) = (\pi) \subset O(U)$.

Suppose $f \in O(U)$. Define the order of vanishing of $f$ along $C$ to be $\text{ord}_C f = \max_n \{ f \in (\pi^n) \}$. Recall from the qualifying exam that $\cap_n (\pi^n) = 0$, so $\text{ord}_C f$ is well-defined.

If $f \in k(X)$, then $f = \frac{p}{q}$ where $p, q \in O(U)$ and $\text{ord}_C f = \text{ord}_C p - \text{ord}_C q$. We define the principal divisor of $f$ to be

$$(f) = \sum_{\text{irreducible curves } C_i} \text{ord}_{C_i}(f) C_i.$$  

Note that this is a finite sum, and is well-defined if $X$ has isolated singularities, i.e. $X^\text{sm} = \{p_1, ..., p_r\}$. In particular, if $X$ is smooth we can define the canonical divisor $K_X = (\omega)$ where $\omega$ is a meromorphic 2-form. If $X$ has isolated singularities, then we can define $K_X^\text{sm} = (\omega)$, where $\omega$ is meromorphic on $X^\text{sm}$ and then close it at $X$ to get $K_X$.

Given $f \in k(X)$ such that $(f) \geq 0$, the divisor $(f)$ is called an effective divisor. Does this imply that $f$ is regular? Not necessarily. Consider the variety $X = \{ e_1, e_2 \} \cup \{ e_3, e_4 \} \subset A^4$. Then $X^\text{sing} = \{0\}$. Choose $f$ such that $f|_{<e_1, e_2>} = 0$ and $f|_{<e_3, e_4>} = 1$. 

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Then \((f)\) is an effective divisor on \(X\), but \(f\) is not a regular function on \(X\). To fix this we have the following

**Definition 1.1.** \(X\) is a normal surface if \(\mathcal{O}_{X,x}\) is integrally closed in \(k(X)\) for all \(x \in X\).

In particular, if \(X\) is normal then

1. \(X^{\text{sing}} = \{p_1, \ldots, p_r\}\) and
2. (Hartog’s Principal) if \((f) \geq 0\) then \(f\) is regular.

**Definition 1.2.** A Cartier or locally principal divisor on \(X\) is a covering \(X = \bigcup U_i\) together with functions \(\{f_i\} \subset k(X)\) such that \((f_i) \subset U_i\) and \(\frac{f_i}{f_j} \in \mathcal{O}^*(U_i \cap U_j)\) where \(\mathcal{O}^*(U_i \cap U_j)\) is the ring of invertible regular functions on \(U_i \cap U_j\).

A Weil divisor is a divisor of the form \(\sum a_iC_i\) where \(a_i \in \mathbb{Z}\) and \(C_i\) are prime divisors.

Note that if \(X\) is smooth, then Weil divisors are Cartier divisors. In general we have

\[
\text{(Principal Divisors)} \subset \text{(Cartier Divisors)} \subset \text{(Weil Divisors)}
\]

We define the **Picard Group** of \(X\) to be

\[
\text{Pic}(X) = \frac{\text{Cartier divisors}}{\text{Principal divisors}}
\]

and the **Divisor Class Group** of \(X\) to be

\[
\text{Cl}(X) = \frac{\text{Weil divisors}}{\text{Principal divisors}}.
\]

**Example 1.3.** Consider the surface \(S = \{z^2 = x^2 + y^2\} \subset \mathbb{R}^2\). Let \(R\) be a ruling of \(S\). Then \(R \subset S\) is a prime divisor, but is not Cartier. However, the divisor \(2R\) is Cartier. In fact, \(2R = (f)\) where \(f\) is the equation of the tangent plane along \(R\). Thus, \(\text{Pic}(X) = 0\), but \(\text{Cl}(X) = \mathbb{Z}/2\mathbb{Z}\).

**Definition 1.4.** A divisor \(D\) is called \(\mathbb{Q}\)-Cartier if \(mD\) is Cartier for some integer \(m > 0\). (In the example above, \(R\) is \(\mathbb{Q}\)-Cartier.) If \(K_X\) is Cartier, then \(X\) is called Gorenstein. If \(K_X\) is \(\mathbb{Q}\)-Cartier, then \(X\) is called \(\mathbb{Q}\)-Gorenstein.

### 2 Linear Systems

Suppose \(X\) is a projective irreducible surface. Let \(D\) be a Cartier divisor. Define

\[
\mathcal{L}(D) = \{f \in k(X)|(f) + D \geq 0\}.
\]

And let \(l(D) = \dim \mathcal{L}(D) < \infty\). The **Linear system** of \(D\) is

\[
|D| = \{(f) + D|(f) \in \mathcal{L}(D)\}.
\]
Choose a linearly independent subset $f_0, \ldots, f_r$ of $\mathcal{L}(D)$. Then there exists a rational map $\phi_D : X \dashrightarrow \mathbb{P}^r$ given by $x \mapsto [f_0(x) : \cdots : f_r(x)]$. If the subset $f_0, \ldots, f_r$ is in fact a basis of $\mathcal{L}(D)$, then the map $\phi_D$ is called a complete linear system. Note that the indeterminacy locus consists of a bunch of points. If the subset $f_0, \ldots, f_r$ is not a basis of $\mathcal{L}(D)$, then the map $\phi_D$ is called a complete linear system.

On the other hand, given any rational map $f : X \dashrightarrow \mathbb{P}^r$, we can write $f = [f_0 : \cdots : f_r]$. Let $D$ be the common denominator of the $f_i$’s. Then by definition, we see that $f_i \in \mathcal{L}(D)$ for all $i$. Thus, any rational map is given by an incomplete linear system, and the divisors $(f_0) + D, \ldots, (f_r) + D$ are pull-backs of coordinate hyperplanes.

Let $D$ be an effective Cartier divisor and consider $|D| = \{\text{effective divisors linearly equivalent to } D\}$. (Recall that $D \sim D'$ if $D - D' = (f)$ for some $f \in k(X)$.) We can decompose $D$ into $D = F + M$ where $F$ is the “fixed part” (i.e. if $D' \in |D|$ then $D' \geq F$) and $M$ is the “moving divisor.” Then $|D| = F + |M|$ and so $\phi_D = \phi_M$.

Let $D$ be a divisor on a surface $S$. Then $l(D) = h^0(D)$. There are numbers $h^1(D)$ and $h^2(D)$, the dimensions of the cohomology groups. Serre’s Duality tells us that $h^2(D) = h^0(K - D)$, where $K$ is the canonical divisor of $S$, and also that $h^1(D) = h^1(K - D)$.

**Definition 2.1.** The Euler Characteristic is $\chi(D) = h^0(D) - h^1(D) + h^2(D)$.

**Theorem 2.2** (Riemann-Roch). $\chi(D) = \chi(0) + \frac{D(D-K)}{2}$ where here $\cdot$ is the intersection product.

**Theorem 2.3** (Noether’s Formula). $\chi_{hol} = \chi(0) = \frac{K_S^2 + \pi}{12}$ where $\pi$ is the topological Euler characteristic.

### 3 Surfaces of General Type

**Definition 3.1.** A divisor $D$ is called very ample if the map $\phi_D : X \hookrightarrow \mathbb{P}^r$ is an embedding. A divisor $D$ is called ample if for some $m > 0$, the divisor $mD$ is very ample.

Some Examples:

1. Let $S \subset \mathbb{P}^3$ be a quintic surface. By the adjunction formula, we have $K_S = H \cap S$, where $H$ is the hyperplane class of $S$. Thus, $\phi_{K_S} : S \hookrightarrow \mathbb{P}^3$ is an embedding. In particular, $K$ is very ample.

2. Suppose $f : S \to \mathbb{P}^2$ is a 2:1 map ramified along an octic $C$. Then $K_S = f^*L$ where $L \subset \mathbb{P}^2$ is a line. And $\phi_K$, the “canonical map” is the same as $f$. So in this case, $K$ is not very ample, but since $2K$ is very ample $K$ is ample.

3. Consider a 2:1 map $f : S \to \mathbb{P}^2$, whose branch locus is a degree 10 curve. Then $K_S = f^*(2L)$, and $\phi_{K_S} : S \to \mathbb{P}^5$ is the composition of $f$ and the
Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$. Thus, $K_S$ is not very ample, but since $2K$ is very ample, $K$ is ample.

Let $X$ be a smooth projective surface. We now have “pluricanonical maps” $\phi_{nK} : X \dashrightarrow \mathbb{P}^N$. We define the Kodaira dimension $\kappa(X) = \sup \dim \phi_{nK}(X)$. If $K$ is ample, then $\kappa(X) = 2$.

**Definition 3.2.** $X$ is a surface of general type if $\kappa(X) = 2$.

In particular, the examples above are surfaces of general type.

How can we tell whether or not a divisor is ample?

**The idea:** If $D$ is ample, then $nD$ is a hyperplane section in some projective embedding. Thus $D \cdot C > 0$ for any irreducible curve $C$. Consider the Neron-Severi subspace $NS \subset H^2(X, \mathbb{R})$, i.e. the subspace spanned by algebraic curves.

The Hodge Index Theorem states that the intersection pairing has signature $(+, -, -, \ldots)$ on $NS$. The Nori cone, i.e., the cone spanned by algebraic curves, is a subset of $NS$.

**Theorem 3.3** (Kleiman’s Criterion). $D$ is ample if and only if

$$NE \setminus \{0\} \subset \{ C | D \cdot C > 0 \}$$

Suppose $S$ is a surface of general type. If $K \cdot C < 0$ for some algebraic curve $C$, then $S$ contains a $(-1)$-curve $C'$, i.e. $C' \cong \mathbb{P}^1$, $C'^2 = -1$, and $K \cdot C' = -1$.

**Theorem 3.4** (Castelnuovo’s Criterion). Any $(-1)$-curve can be contracted, i.e., there exists a map from $S$ to a surface $S'$ such that $S'$ is of general type.

**Definition 3.5.** $X$ is called minimal if it has no $(-1)$-curves.

So any surface of general type is an iterated blow-up of a minimal surface of general type.

Questions:
1. Classify the numerical invariants (pairs $(\kappa^2, \chi_{\text{top}})$) of minimal surfaces.
2. Fix numerical types and understand the moduli space.

**Theorem 3.6** (Gieseker). Let $X$ be a smooth minimal surface of general type. The $\phi_{5K} : X \to \mathbb{P}^r$ is birational onto its image and $\phi_{5K}(X)$ has duVal singularities. $\phi_{5K}$ is a “canonical model.”