(1) Let \( f : X \to Y \) be a non constant holomorphic map between compact Riemann surfaces. Show that \( f \) is surjective using point set topology. [Hint: \( f \) is an open mapping.]

(2) Recall Liouville’s theorem: a bounded holomorphic function \( f : \mathbb{C} \to \mathbb{C} \) is constant. Deduce that a global holomorphic function \( f : \mathbb{P}^1 \to \mathbb{C} \) is constant. (Reminder: We showed that a holomorphic function on any compact complex manifold is constant using the maximum principle. This exercise gives an alternative argument in the case of \( \mathbb{P}^1 \).)

(3) Recall that a hypersurface

\[
X = (F(X_0, \ldots, X_n) = 0) \subset \mathbb{P}^n
\]

is smooth (that is, a complex manifold) iff \( \frac{\partial F}{\partial X_i}(P) \neq 0 \) for some \( i \) for each \( P \in X \). (This follows from the inverse function theorem.)

(a) Show that \( \sum \frac{\partial F}{\partial X_i} \cdot X_i = d \cdot F \) where \( d \) is the degree of \( F \). So the above condition is equivalent to \( \frac{\partial F}{\partial X_i}(P) \neq 0 \) for some \( i \) for each \( P \in \mathbb{P}^n \).

(b) Show that the tangent plane at a point \( P \in X \) is given by the hyperplane

\[
\left( \sum \frac{\partial F}{\partial X_i}(P) \cdot X_i = 0 \right) \subset \mathbb{P}^n.
\]

(c) Show that the Fermat hypersurface \( X = (X_0^d + \cdots + X_n^d = 0) \subset \mathbb{P}^n \) is smooth.

(4) Let \( X \subset \mathbb{P}^n \) be a quadric (a hypersurface of degree 2).

(a) Show that, after a change of coordinates, \( X \) is given by

\[
X = (X_0^2 + \cdots + X_m^2 = 0) \subset \mathbb{P}^m
\]
where \( m \leq n \). [Hint: This is a question about quadratic forms.]

(b) Show that \( X \) is smooth iff \( m = n \).

(c) If \( m = n = 2 \) show that after a change of coordinates

\[
X = (X_0X_2 = X_1^2) \subset \mathbb{P}^2.
\]

Show that there is an isomorphism

\[
\mathbb{P}^1 \to X
\]

given by

\[
(Y_0 : Y_1) \mapsto (Y_0^2 : Y_0Y_1 : Y_1^2).
\]

(d) If \( m = n = 3 \) show that after a change of coordinates

\[
X = (X_0X_3 = X_1X_2) \subset \mathbb{P}^3.
\]

Show that there is an isomorphism

\[
\mathbb{P}^1 \times \mathbb{P}^1 \to X
\]

given by

\[
((Y_0 : Y_1), (Z_0 : Z_1)) \mapsto (Y_0Z_0 : Y_0Z_1 : Y_1Z_0 : Y_1Z_1).
\]

(5) Let \( f: X \to Y \) be a non constant holomorphic map between compact Riemann surfaces. Show that \( g(X) \geq g(Y) \), and that if \( g(X) = g(Y) \) then either \( g(Y) = 1 \) and \( f \) is unramified or \( g(Y) = 0 \). Give examples to show that these cases do occur. [Hint: A Riemann surface \( Y \) of genus 1 is a complex torus \( \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau) \), some \( \tau \in \mathbb{C} \setminus \mathbb{R} \). The only Riemann surface of genus 0 is \( \mathbb{P}^1_{\mathbb{C}} \).]

(6) Recall that if \( a_1, \ldots, a_{2g+2} \) are distinct points in \( \mathbb{P}^1_{\mathbb{C}} \) we constructed a compact Riemann surface \( X \) of genus \( g \) and a degree 2 map \( f: X \to \mathbb{P}^1 \) branched over the \( a_i \). (\( X \) is a so called hyperelliptic Riemann surface.) What happens if we allow two of the \( a_i \) to come together? Draw a picture showing the topology change.

(7) Let \( P \in \mathbb{P}^n \) be a point, and let \( \pi \) be the projection

\[
\pi: \mathbb{P}^n \to \mathbb{P}^{n-1}
\]

given by sending a point \( Q \in \mathbb{P}^n \setminus \{P\} \) to the line \( PQ \) in the projective space \( \mathbb{P}(T_P\mathbb{P}^n) \) of the tangent space to \( \mathbb{P}^n \) at \( P \). (We use a dashed arrow because \( \pi \) is not defined at \( P \).) Show that if \( P = (0 : 0 : \cdots : 0 : 1) \) then \( \pi \) is given in coordinates by

\[
(X_0 : \cdots : X_n) \mapsto (X_0 : \cdots : X_{n-1}).
\]
(8) Recall that if \( X \subset \mathbb{P}^2 \) is a plane curve of degree \( d \) then the projection \( \pi : \mathbb{P}^2 \to \mathbb{P}^1 \) from a point \( P \in \mathbb{P}^2 \setminus X \) induces a map \( X \to \mathbb{P}^1 \) of degree \( d \). Now suppose we project from a point \( P \in X \). Show that we obtain a map \( f : X \to \mathbb{P}^1 \) of degree \( d - 1 \) which is well defined at \( P \) (even though \( \pi \) is not). What is \( f(P) \)?

(9) Let \( X \subset \mathbb{P}^2 \) be a plane curve. We say \( X \) has a node at \( P \) if there are local coordinates \( x, y \) at \( P \in \mathbb{P}^2 \) such that \( X = (xy = 0) \) near \( P \). (This is the simplest possible type of singularity for a curve.) Now suppose \( X \) has degree \( d \), has \( \delta \) nodes, and is smooth elsewhere. Let \( \tilde{X} \) be the Riemann surface obtained from \( X \) by separating the two smooth branches at each node. Show that

\[
g(X) = \frac{1}{2} (d-1)(d-2) - \delta.
\]

[Hint: Generalize the argument we used in the smooth case.]

(10) If \( X \) is a compact Riemann surface then there exists an embedding \( X \subset \mathbb{P}^N \) in projective \( N \)-space for some \( N \) (this is a hard theorem). Show that if \( N \geq 4 \) there is a projection \( \pi : \mathbb{P}^N \to \mathbb{P}^{N-1} \) which induces an embedding \( X \subset \mathbb{P}^{N-1} \). [Hint: Show that the locus of points in \( \mathbb{P}^N \) for which the projection does not yield an embedding has dimension \( \leq 3 \). So any compact Riemann surface can be embedded in \( \mathbb{P}^3 \). Use the same argument to show that any Riemann surface admits an immersion in \( X \to \mathbb{P}^2 \), that is, a map with image a curve with only nodal singularities.

(11) Let \( G \) be a finite group acting on a compact Riemann surface \( X \).

(a) Let \( P \in X \) be a point. Show that the stabilizer \( G_P \subset G \) of \( P \) is a cyclic group acting by

\[
z \mapsto \zeta z
\]

for \( \zeta \) a root of unity, for some choice of local coordinate \( z \) at \( P \). [Hint: Let \( \chi : G_P \to \mathbb{C}^\times \) be the character of \( G_P \) giving the action of \( G \) on the tangent space at \( P \). Let \( w \) be some local coordinate at \( P \), and define \( z = \frac{1}{|G_P|} \sum_{g \in G_P} \chi(g)^{-1} g^* w \). Show that \( z \) is a local coordinate at \( P \) and \( g^* z = \chi(g) \cdot z \).]

(b) Use part (a) to define the structure of a Riemann surface on the topological space \( Y = X/G \).
(c) Show that

$$2g(X) - 2 = |G| \cdot \left( (2g(Y) - 2) + \sum_{P \in X} \left( 1 - \frac{1}{|G_P|} \right) \right).$$

(d) Deduce that if $g(X) \geq 2$ then $|G| \leq 84(g(X) - 1)$. [Hint: Consider the cases $g(Y) \geq 2$, $g(Y) = 1$, and $g(Y) = 0$ separately. For $g(Y) = 0$ you will need to show the following lemma: if $\{e_P\}$ are positive integers and $S := \sum(1 - \frac{1}{e_P}) \geq 2$, then $S \geq 2 \frac{1}{12}$ (with equality iff $\{e_P\} = \{2, 3, 7\}$).]