EXCEPTIONAL BUNDLES ASSOCIATED TO DEGENERATIONS OF SURFACES

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1. Introduction

In 1981 J. Wahl described smoothings of surface quotient singularities with no vanishing cycles [W81, 5.9.1]. Given a smoothing of a projective surface X of this type, we construct an associated exceptional vector bundle on the nearby fiber Y in the case $H^{2,0}(Y) = H^1(Y) = 0$. If $Y = \mathbb{P}^2$ we show that our construction establishes a bijective correspondence between the possible degenerate surfaces X and exceptional bundles on Y modulo a natural equivalence relation. If Y is of general type then our construction establishes a connection between components of the boundary of the moduli space of surfaces deformation equivalent to Y and exceptional bundles on Y.

Let n, a be positive integers such that $a < n$ and $(a, n) = 1$. Consider the cyclic quotient singularity

$$(0 \in \mathbb{A}^2/(\mathbb{Z}/n^2\mathbb{Z})),\\ \mathbb{Z}/n^2\mathbb{Z} \ni 1\colon (u,v) \mapsto (\xi u, \xi^{na-1} v), \quad \xi = \exp(2\pi i/n^2). \tag{1.1}$$

We refer to (1.1) as a *Wahl singularity* of type $\frac{1}{n^2}(1, na-1)$. A Wahl singularity admits a $\mathbb{Q}$-*Gorenstein smoothing*, that is, a one parameter deformation such that the general fiber is smooth and the canonical divisor of the total space is $\mathbb{Q}$-Cartier. The Milnor fiber of such a smoothing is a rational homology ball. So, if Y is the general fiber of a $\mathbb{Q}$-Gorenstein smoothing of a surface X with Wahl singularities, then the specialization map

$$H_*(Y, \mathbb{Q}) \to H_*(X, \mathbb{Q})$$

is an isomorphism. For this reason, it is difficult to predict the existence of the degeneration $Y \rightsquigarrow X$ given the surface Y.

An *exceptional bundle* F on a projective surface Y is a locally free sheaf such that $\operatorname{Hom}(F, F) = \mathbb{C}$ and $\operatorname{Ext}^1(F, F) = \operatorname{Ext}^2(F, F) = 0$. In particular F is indecomposable, rigid (no infinitesimal deformations), and unobstructed in families. So, if $\mathcal{Y}/(0 \in S)$ is a deformation of Y over a germ $(0 \in S)$, then

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F deforms in a unique way to a family of exceptional bundles on the fibers of $\mathcal{Y}/(0 \in S)$.

Theorem 1.1. *Let X be a projective normal surface with a unique singularity $P \in X$ of Wahl type $\frac{1}{n^2}(1, na-1)$. Let $\mathcal{X}/(0 \in T)$ be a one parameter deformation of X such that the general fiber Y is smooth and the canonical divisor $K_{\mathcal{X}}$ of the total space is $\mathbb{Q}$-Cartier.*

(1) *Assume that $H_1(Y, \mathbb{Z})$ is finite of order coprime to n. Then the specialization map*
$$\mathrm{sp}\colon H_2(Y, \mathbb{Z}) \to H_2(X, \mathbb{Z})$$
is injective with cokernel isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

(2) *Assume in addition that $H^{2,0}(Y) = 0$. Then, after a base change $(0 \in T') \to (0 \in T)$ of ramification index a, there exists a reflexive sheaf $\mathcal{E}$ on $\mathcal{X}' := \mathcal{X} \times_T T'$ such that*
   (a) *$F := \mathcal{E}|_Y$ is an exceptional bundle of rank n on Y, and*
   (b) *$E := \mathcal{E}|_X$ is a torsion-free sheaf on X such that its reflexive hull $E^{\vee\vee}$ is isomorphic to the direct sum of n copies of a reflexive rank 1 sheaf A, and the quotient $E^{\vee\vee}/E$ is a torsion sheaf supported at $P \in X$.*

If $\mathcal{H}$ is a line bundle on $\mathcal{X}/T$ which is ample on the fibers, then F is slope stable with respect to the the ample line bundle $H := \mathcal{H}|_Y$. Moreover, we have
$$c_1(F) = nc_1(A) \in H_2(Y, \mathbb{Z}) \subset H_2(X, \mathbb{Z}),$$
$$c_2(F) = \frac{n-1}{2n}(c_1(F)^2 + n + 1),$$
$$c_1(F) \cdot K_Y = \pm a \mod n,$$
and
$$H_2(X, \mathbb{Z}) = H_2(Y, \mathbb{Z}) + \mathbb{Z} \cdot (c_1(F)/n).$$

Remark 1.2. The torsion-free sheaf E on X is a Gieseker semistable limit of the family of stable exceptional bundles F on the fibers of $\mathcal{X}'/T'$ over $T' \setminus \{0\}$. If E is Gieseker stable, then it is uniquely determined by this property. See [HL97, 2.B.1].

Remark 1.3. The exceptional bundles on Y obtained from F by dualizing or tensoring by a line bundle arise from the degeneration $\mathcal{X}/(0 \in T)$ in the same way. Indeed, the dual $\mathcal{E}^\vee$ of $\mathcal{E}$ satisfies the properties 1.1(2). Similarly, if L is a line bundle on Y, then L extends to a reflexive rank 1 sheaf $\mathcal{L}$ on $\mathcal{X}'$, and the reflexive hull of the tensor product $\mathcal{E} \otimes \mathcal{L}$ satisfies the properties 1.1(2).

Recent work of Y. Lee and J. Park constructs new surfaces of general type with $H^{2,0} = H^1 = 0$ as $\mathbb{Q}$-Gorenstein smoothings of rational surfaces with Wahl singularities, see e.g. [LP07]. In these cases our construction produces examples of exceptional bundles of rank greater than 1 on surfaces of general type. As far as I know these are the first such examples. In general little is known about moduli spaces of stable bundles on surfaces of general type unless the expected dimension is large.

The idea of the proof of Theorem 1.1 is as follows. After a base change (which we suppress in our notation), there is a proper birational morphism

$$\pi\colon \tilde{\mathcal{X}} \to \mathcal{X}/(0 \in T)$$

with exceptional locus a normal surface W such that $\pi(W) = P \in X$. The special fiber $\tilde{X} := \tilde{\mathcal{X}}_0$ is the union of X' (the strict transform of X) and W, with scheme theoretic intersection a smooth rational curve C. The surface W is isomorphic to an explicit weighted projective hypersurface determined by n and a. Our topological assumptions imply that there is a Cartier divisor D' on X' such that $D' \cdot C = 1$. Given an exceptional bundle G on W such that $G|_C \simeq \mathcal{O}_C(1)^{\oplus n}$, we glue G and $\mathcal{O}_{X'}(D')^{\oplus n}$ to obtain an exceptional bundle $\tilde{E}$ on the reducible surface $\tilde{X}$. Since $\tilde{E}$ is exceptional it extends uniquely to a locally free sheaf $\tilde{\mathcal{E}}$ on $\tilde{\mathcal{X}}/(0 \in T)$, and the restriction F of $\tilde{\mathcal{E}}$ to the general fiber Y is exceptional. The exceptional bundle G on $W = W_{n,a}$ is constructed by induction on a, using a degeneration of W to a surface with a Wahl singularity of type $\frac{1}{a^2}(1, ab-1)$, where $b = n \bmod a$.

Notation. We work in the algebraic category over the complex numbers. By a *germ* $(P \in S)$ we mean an étale neighborhood of a point P on an scheme S of finite type over $\mathbb{C}$, it being understood that we may restrict to another étale neighborhood without further comment. By a *one parameter deformation* of a scheme X we mean a flat morphism $\mathcal{X} \to (0 \in T)$ from a scheme $\mathcal{X}$ to a smooth curve germ $(0 \in T)$, together with an identification of X with the special fiber $\mathcal{X}_0$.

In what follows we use the shorthand $\mathbb{A}^d/\frac{1}{r}(a_1, \ldots, a_d)$ or just $\frac{1}{r}(a_1, \ldots, a_d)$ for the cyclic quotient

$$\mathbb{A}^d/(\mathbb{Z}/r\mathbb{Z}),$$

$$\mathbb{Z}/r\mathbb{Z} \ni 1\colon (x_1, \ldots, x_d) \mapsto (\zeta^{a_1}x_1, \ldots, \zeta^{a_d}x_d), \quad \zeta = \exp(2\pi i/r).$$

Some background on reflexive sheaves and toric geometry is reviewed in §7.

2. Wahl singularities

Let $(P \in X)$ denote the Wahl singularity $(0 \in \mathbb{A}^2_{u,v}/\frac{1}{n^2}(1, na-1))$. The canonical divisor K_X is $\mathbb{Q}$-Cartier of index n, that is, nK_X is Cartier and $n \in \mathbb{N}$ is minimal with this property. Thus K_X defines a cyclic covering

$$\pi\colon (Q \in Z) \to (P \in X)$$

of degree n, which is unramified over $X \setminus \{P\}$, such that $K_Z = \pi^* K_X$ is Cartier. The covering π is called the *index one cover*. It is uniquely determined locally for the étale topology at $P \in X$. Explicitly, we have

$$Z = \mathbb{A}^2_{u,v}/\tfrac{1}{n}(1, -1) = (xy = z^n) \subset \mathbb{A}^3_{x,y,z}$$

where $x = u^n, y = v^n, z = uv$. Thus

$$X = (xy = z^n) \subset \left(\mathbb{A}^3_{x,y,z}/\tfrac{1}{n}(1,-1,a)\right).$$

A smoothing of $(P \in X)$ is given by

$$\mathcal{X} = (xy = z^n + t) \subset \left(\mathbb{A}^3_{x,y,z}/\tfrac{1}{n}(1,-1,a)\right) \times \mathbb{A}^1_t. \tag{2.1}$$

The link L of the singularity $(P \in X)$ is the lens space $S^3/\frac{1}{n^2}(1, na-1)$. Let M denote the Milnor fiber of the smoothing (2.1), a smooth 4-manifold with boundary L. (See e.g [L84, 2.B] for the definition and basic properties of the Milnor fiber of the smoothing of an isolated singularity.)

Lemma 2.1. [W81, 5.9.1] *The Milnor fiber M is a rational homology ball. More precisely, $\pi_1(M) = \mathbb{Z}/n\mathbb{Z}$, $H_i(M,\mathbb{Z}) = 0$ for $i > 1$, and the map $\pi_1(L) \to \pi_1(M)$ is the quotient map $\mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$.*

Proof. Note that by construction M is the quotient of the Milnor fiber M_Z of a smoothing of $(Q \in Z)$, a Du Val singularity of type A_{n-1}, by a free action of $\mathbb{Z}/n\mathbb{Z}$. The Milnor fiber M_Z has the homotopy type of a bouquet of $n-1$ copies of S^2. So in particular M_Z is simply connected and $\pi_1(M) = \mathbb{Z}/n\mathbb{Z}$. Since M is Stein of complex dimension 2 it has the homotopy type of a cell complex of real dimension 2. Finally the Euler number $e(M) = e(M_Z)/n = 1$, so $b_2(M) = 0$.

The map $\pi_1(L) \to \pi_1(M)$ is surjective because the inverse image of L in the universal cover M_Z of M is connected. Writing $L = S^3/(\mathbb{Z}/n^2\mathbb{Z})$ and $M = M_Z/(\mathbb{Z}/n\mathbb{Z})$ as above, the map $\pi_1(L) \to \pi_1(M)$ is identified with the quotient map $\mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. □

Remark 2.2. A more explicit analysis yields the following topological description of M, see [K92, 2.1]. Let N_Z be the topological space obtained from n copies Δ_j of the closed disc $\Delta := (|z| \leq 1) \subset \mathbb{C}$ by identifying their boundaries. Define a free action of $\mathbb{Z}/n\mathbb{Z}$ on N_Z by

$$\Delta_j \to \Delta_{j+1}, \quad z \mapsto \zeta z, \quad \zeta = \exp(2\pi i/n),$$

where the indices j are understood modulo n. Let N denote the quotient $N_Z/(\mathbb{Z}/n\mathbb{Z})$. Then M is homotopy equivalent to N.

The $\mathbb{Q}$-Gorenstein deformations of a quotient singularity are by definition those deformations induced by an equivariant deformation of the index one cover [H04, 3.1]. A one parameter smoothing of a quotient singularity is $\mathbb{Q}$-Gorenstein iff the canonical divisor of the total space is $\mathbb{Q}$-Cartier [H04, 3.4]. The deformation (2.1) is a versal $\mathbb{Q}$-Gorenstein deformation, that is, every $\mathbb{Q}$-Gorenstein deformation of $(P \in X)$ is obtained from (2.1) by pullback. Indeed, the versal deformation of the index one cover $(Q \in Z)$ is given by

$$(xy = z^n + a_{n-2}z^{n-2} + \cdots + a_1 z + a_0) \subset \mathbb{A}^3_{x,y,z} \times \mathbb{A}^{n-1}_{a_0,\ldots,a_{n-2}}.$$

The $\mathbb{Z}/n\mathbb{Z}$-equivariant deformations of $(Q \in Z)$ are given by the locus

$$\mathbb{A}^1_{a_0} = (a_1 = a_2 = \cdots = a_{n-2} = 0) \subset \mathbb{A}^{n-1}_{a_0,\ldots,a_{n-2}}.$$

The $\mathbb{Q}$-Gorenstein condition is natural from the point of view of Mori theory and is used in the definition of the compactification $\overline{\mathcal{M}}$ of the moduli space of surfaces of general type analogous to the Deligne–Mumford compactification of the moduli space of curves [KSB88]. The following observation is a key motivation for our paper.

Lemma 2.3. *Let X be a normal projective surface such that K_X is ample, X has a unique singularity of Wahl type, and $H^2(T_X) = 0$. Then the Kollár–Shepherd-Barron moduli stack $\overline{\mathcal{M}}$ of stable surfaces is smooth near $[X]$ and locally trivial deformations of X determine a codimension one component of the boundary of $\overline{\mathcal{M}}$.*

Proof. Let $\mathcal{X}/(0 \in D)$ be the versal $\mathbb{Q}$-Gorenstein deformation of X, $(P \in \mathcal{X}_{\text{loc}})/(0 \in C)$ the versal $\mathbb{Q}$-Gorenstein deformation of the Wahl singularity $(P \in X)$, and let $f\colon (0 \in D) \to (0 \in C)$ be a morphism such that the deformation $(P \in \mathcal{X})/(0 \in D)$ of $(P \in X)$ is isomorphic to the pullback of $(P \in \mathcal{X}_{\text{loc}})/(0 \in C)$. (Note that both versal deformations are algebraizable: indeed $\mathcal{X}/(0 \in D)$ is polarized by the relative dualizing sheaf $\omega_{\mathcal{X}/D}$ so we can apply the Grothendieck existence theorem [G61, 5.4.5], and deformations of isolated singularities are algebraizable by [E73]. So we may assume that $f\colon (0 \in D) \to (0 \in C)$ is a morphism of schemes of finite type over $\mathbb{C}$). The morphism f is smooth because $H^2(T_X) = 0$ (see Lemma 7.2). The space $(0 \in C)$ is smooth of dimension 1 and the general fiber of the deformation $(P \in \mathcal{X}_{\text{loc}})/(0 \in C)$ is smooth (see §2). In particular $(0 \in D)$ is smooth, and the locus $H = f^{-1}(0) \subset D$ of locally trivial deformations of X is smooth of codimension 1.

Let $\overline{M}$ denote the coarse moduli space of the Deligne–Mumford stack $\overline{\mathcal{M}}$ of stable surfaces. Étale locally over $[X] \in \overline{M}$, the stack $\overline{\mathcal{M}}$ is identified with the quotient stack $[(0 \in D)/G]$ where $G = \text{Aut}(X)$ (a finite group). Thus the stack $\overline{\mathcal{M}}$ is smooth at $[X] \in \overline{\mathcal{M}}$, and near $[X]$ the boundary $\partial\overline{\mathcal{M}}$ of $\overline{\mathcal{M}}$ (the locus of singular surfaces) is given by the smooth divisor $H \subset D$. Thus $\partial\overline{\mathcal{M}} \subset \overline{\mathcal{M}}$ is smooth of codimension 1 at $[X] \in \overline{\mathcal{M}}$. □

3. Blowup construction

Proposition 3.1. *Let n and a be positive integers such that $a < n$ and $(a, n) = 1$. Let $(P \in \mathcal{X})/(0 \in T)$ be a one parameter $\mathbb{Q}$-Gorenstein smoothing of a Wahl singularity $(P \in X) \simeq (0 \in \mathbb{A}^2_{u,v}/\frac{1}{n^2}(1, na-1))$. Then, after a base change $(0 \in T') \to (0 \in T)$ of ramification index a, there exists a birational morphism $\pi\colon \tilde{\mathcal{X}} \to \mathcal{X}'$ satisfying the following properties.*

1. *The locus $W := \pi^{-1}(P) \subset \tilde{\mathcal{X}}$ is a normal surface isomorphic to the weighted projective hypersurface*

$$(XY = Z^n + T^a) \subset \mathbb{P}(1, na-1, a, n).$$

Moreover, $W \subset \tilde{\mathcal{X}}$ is a $\mathbb{Q}$-Cartier divisor.

2. *The morphism π restricts to an isomorphism $\tilde{\mathcal{X}} \setminus W \to \mathcal{X} \setminus \{P\}$.*

(3) *The special fiber* $\tilde{X} := \tilde{\mathcal{X}}_0$ *is reduced and is the union of two components* $\tilde{X}_1$ *and* $\tilde{X}_2$ *meeting along a smooth rational curve* C*, where* $\tilde{X}_1$ *is the strict transform of* X *and* $\tilde{X}_2 = W$ *is the exceptional divisor. The curve* $C \subset \tilde{X}_1$ *is the exceptional curve of the restriction* $p\colon \tilde{X}_1 \to X$ *of* π*, and* $C = (T = 0) \subset W$*.*

(4) *Let* $Q \in C \subset \tilde{X}$ *denote the point with homogeneous coordinates* $(0\colon 1\colon 0\colon 0)$*. The reducible surface* $\tilde{X}$ *has normal crossing singularities* $(xy = 0) \subset \mathbb{A}^3_{x,y,z}$ *along* $C \setminus \{Q\}$*, an orbifold normal crossing singularity* $(xy = 0) \subset \mathbb{A}^3_{x,y,z}/\frac{1}{na-1}(1, -1, a^2)$ *at* Q*, and is smooth elsewhere.*

(5) *The birational morphism* $p\colon \tilde{X}_1 \to X$ *is the weighted blowup of* $P \in X$ *with weights* $\frac{1}{n^2}(1, na-1)$ *with respect to the orbifold coordinates* u, v*. In particular the strict transform* D' *of* $D = (v = 0) \subset (P \in X)$ *is a Cartier divisor such that* $D' \cdot C = 1$*.*

Proof. Recall from §2 that the versal $\mathbb{Q}$-Gorenstein deformation of $(P \in X)$ is given by

$$\mathcal{X}^{\text{ver}} = (xy = z^n + t) \subset \left(\mathbb{A}^3_{x,y,z}/\tfrac{1}{n}(1, -1, a)\right) \times \mathbb{A}^1_t.$$

We describe the construction of π for the versal deformation. In general we obtain the morphism by pullback from the versal case. (Note that all the assertions are preserved under pullback. Indeed, let $g\colon (0 \in T) \to (0 \in \mathbb{A}^1_t)$ be a morphism such that the deformation $(P \in \mathcal{X})/(0 \in T)$ of $(P \in X)$ is isomorphic to the pullback of $(P \in \mathcal{X}^{\text{ver}})/(0 \in \mathbb{A}^1_t)$. Let $(W^{\text{ver}} \subset \tilde{\mathcal{X}}^{\text{ver}})$ be a blowup of the base change of $(P \in \mathcal{X}^{\text{ver}})$ as in the statement, and $(W \subset \tilde{\mathcal{X}})$ its pullback under g, with induced map $g_{\tilde{\mathcal{X}}}\colon \tilde{\mathcal{X}} \to \tilde{\mathcal{X}}^{\text{ver}}$. Then $W = \frac{1}{l} g^*_{\tilde{\mathcal{X}}} W^{\text{ver}}$ where l is the ramification index of g. Thus W is $\mathbb{Q}$-Cartier because W^{ver} is $\mathbb{Q}$-Cartier. The remaining assertions concern the special fiber and so are preserved under pullback.)

We make the base change $t \mapsto t^a$ and blowup (x, y, z, t) with weights

$$w = \tfrac{1}{n}(1, na-1, a, n) \in \mathbb{Z}^4 + \mathbb{Z} \cdot \tfrac{1}{n}(1, -1, a, 0)$$

to obtain the desired birational morphism. (See §7.2.2 for background on weighted blowups.) Let $f\colon \tilde{\mathbb{A}} \to \mathbb{A}$ denote the blowup of the ambient space $\mathbb{A} := \frac{1}{n}(1, -1, a) \times \mathbb{A}^1_t$. Then f has exceptional divisor

$$E = \mathbb{P}(1, na-1, a, n)$$

with weighted homogeneous coordinates X, Y, Z, T corresponding to the orbifold coordinates x, y, z, t at $0 \in \mathbb{A}$. We define the 3-fold $\tilde{\mathcal{X}}$ to be the strict transform of $\mathcal{X} \subset \mathbb{A}$ under the map f and the morphism $\pi\colon \tilde{\mathcal{X}} \to \mathcal{X}$ to be the morphism induced by f. Observe that the equation

$$(xy = z^n + t^a) \subset \mathbb{A}$$

of $\mathcal{X}$ is homogeneous with respect to the weight vector w. It follows that the exceptional locus

$$W := \pi^{-1}(P) = E \cap \tilde{\mathcal{X}}$$

of π is given by the same equation in the weighted projective space E. Note that $W = E|_{\tilde{\mathcal{X}}}$ is a $\mathbb{Q}$-Cartier divisor on $\tilde{\mathcal{X}}$ because E is $\mathbb{Q}$-Cartier on $\tilde{\mathbb{A}}$. Now consider the fiber $\tilde{X} = \tilde{\mathcal{X}}_0$ of $\tilde{\mathcal{X}}$ over $0 \in T = \mathbb{A}^1_t$. Observe that the weight $w(t)$ of t for the blowup is equal to 1 (we made the base change above to ensure this). It follows that the Cartier divisor $\tilde{X} = (t = 0) \subset \tilde{\mathcal{X}}$ is reduced, equal to the sum $X' + W$ of the strict transform of X and the exceptional divisor W. It is easy to check in the charts for $\tilde{\mathbb{A}}$ that the singularities of $\tilde{X}$ are as described in the statement.

Finally, recalling that $(P \in X) = (t = 0) \subset (P \in \mathcal{X})$ is identified with the cyclic quotient singularity $(0 \in \mathbb{A}^2_{u,v}/\frac{1}{n^2}(1, na-1))$ by

$$(u, v) \mapsto (x, y, z) = (u^n, v^n, uv),$$

a toric computation shows that the induced birational morphism $p\colon \tilde{X}_1 \to X$ is the blowup with weights $\frac{1}{n^2}(1, na-1)$ with respect to u, v. Indeed, write

$$\mathbb{A}_0 := \mathbb{A}^3_{x,y,z}/\tfrac{1}{n}(1, -1, a) = (t = 0) \subset \mathbb{A},$$

an affine toric variety. Let $\tilde{\mathbb{A}}_0$ denote the strict transform of $\mathbb{A}_0$ under f, then the induced morphism $\tilde{\mathbb{A}}_0 \to \mathbb{A}_0$ is the weighted blowup of $\mathbb{A}_0$ with weights $\frac{1}{n}(1, na-1, a)$. The subvariety $(P \in X) \subset (0 \in \mathbb{A}_0)$ is the closure of the subtorus corresponding to the primitive sublattice

$$H := (u_1 + u_2 = nu_3) \subset \mathbb{Z}^3 + \mathbb{Z} \cdot \tfrac{1}{n}(1, -1, a)$$

of the lattice of one parameter subgroups of the big torus of $\mathbb{A}_0$. It follows that the normalization $\tilde{X}_1^\nu$ of the strict transform $\tilde{X}_1 \subset \tilde{\mathbb{A}}_0$ of X under f is given by the toric surface with fan the intersection of the fan of $\tilde{\mathbb{A}}_0$ with the subspace $H \otimes \mathbb{R}$. Computing this fan explicitly shows that $\tilde{X}_1^\nu$ is the weighted blowup of $X = (0 \in \mathbb{A}^2/\frac{1}{n^2}(1, na-1))$ with weights $\frac{1}{n^2}(1, na-1)$. Finally, working in charts for $\tilde{\mathbb{A}}_0$, we see that $\tilde{X}_1$ is normal (in each chart it is a quotient of a smooth hypersurface in $\mathbb{A}^3$).

The surface $\tilde{X}_1$ is smooth along D' because the toric chart for $\tilde{X}_1$ meeting D' corresponds to a cone of the fan generated by a basis of the lattice. Thus D' is Cartier. Moreover D' and C are given by the two toric boundary divisors in this chart, which intersect transversely in a single smooth point. So $D' \cdot C = 1$.

We remark that, since $P \in X$ is a cyclic quotient singularity, its minimal resolution $\hat{X} \to X$ has exceptional locus a nodal chain $F = F_1 + \ldots + F_r$ of smooth rational curves, such that the strict transforms $(u = 0)'$ and $(v = 0)'$ of the coordinate axes intersect the end components F_1 and F_r respectively. Then $\tilde{X}_1$ is obtained from $\hat{X}$ by contracting the chain $F_1 + \cdots + F_{r-1}$ of exceptional curves disjoint from $(v = 0)'$ to a cyclic quotient singularity of type $\frac{1}{na-1}(a^2, -1)$. (This can be verified using the description of the weighted blowup in §7.2.2 and the toric construction of the minimal resolution of a cyclic quotient surface singularity described in [F93, 2.6].) □

4. Glueing

Let $\mathcal{X}/(0 \in T)$ be a one parameter deformation of a projective normal surface X with quotient singularities. Let $P \in X$ be a Wahl singularity of type $\frac{1}{n^2}(1, na-1)$ such that the germ $(P \in \mathcal{X})/(0 \in T)$ is a $\mathbb{Q}$-Gorenstein smoothing of $(P \in X)$. Let $P_1 = P, P_2, \ldots, P_r$ be the singularities of X and L_i the link of the singularity $P_i \in X$. Let Y denote a general fiber of $\mathcal{X}/T$. In this section we make the following assumptions:

(1) The map

$$H_2(X, \mathbb{Z}) \to \bigoplus H_1(L_i, \mathbb{Z}), \quad \alpha \mapsto (\alpha \cap L_i) \tag{4.1}$$

is surjective.

(2) We have $H^2(\mathcal{O}_Y) = 0$ and $H^1(Y, \mathbb{Z}) = 0$.

Let $(0 \in T') \to (0 \in T)$ and $\pi \colon \tilde{\mathcal{X}} \to \mathcal{X}'$ be the base change and blowup of Proposition 3.1.

Lemma 4.1. *There exists a line bundle $\tilde{A}$ on the strict transform $\tilde{X}_1 \subset \tilde{\mathcal{X}}$ of X such that the restriction of $\tilde{A}$ to the exceptional curve C of $p \colon \tilde{X}_1 \to X$ has degree 1. Moreover, $H^i(\mathcal{O}_{\tilde{X}_1}) = 0$ for $i > 0$.*

Proof. By our assumptions $H^i(\mathcal{O}_Y) = 0$ for $i > 0$. Since X has quotient singularities, we have $H^i(\mathcal{O}_{\tilde{X}_1}) = H^i(\mathcal{O}_X)$ (quotient singularities are rational) and $H^i(\mathcal{O}_X) = H^i(\mathcal{O}_Y)$ (quotient singularities are Du Bois [DB81, 4.6,5.3]). Thus $H^i(\mathcal{O}_{\tilde{X}_1}) = H^i(\mathcal{O}_X) = 0$ for $i > 0$. In particular, $c_1 \colon \mathrm{Cl}(X) \to H_2(X, \mathbb{Z})$ is an isomorphism (see §7.1), and the map 4.1 is identified with the local-to-global map for the class group of X

$$\mathrm{Cl}(X) \to \bigoplus \mathrm{Cl}(P_i \in X).$$

By surjectivity of (4.1), there exists an effective Weil divisor $D \subset X$ such that D is given by the zero locus of the orbifold coordinate v at $(P \in X)$ for some identification

$$(P \in X) \simeq (0 \in \mathbb{C}^2_{u,v}/\tfrac{1}{n^2}(1, na-1)),$$

and D does not pass through the remaining singularities of X. Then D' is a Cartier divisor on $\tilde{X}_1$ such that $D' \cdot C = 1$ by Proposition 3.1(5). So we may take $A = \mathcal{O}_{\tilde{X}_1}(D')$. □

Proposition 4.2. *Suppose G is an exceptional bundle of rank n on the π-exceptional divisor W such that $G|_C \simeq \mathcal{O}_C(1)^{\oplus n}$. Let $\tilde{E}$ be the vector bundle on the reducible surface $\tilde{X}$ obtained by glueing $\tilde{A}^{\oplus n}$ on $\tilde{X}_1$ and G on $\tilde{X}_2 = W$ along $\mathcal{O}_C(1)^{\oplus n}$ on C (see Lemma 7.3). Then $\tilde{E}$ is an exceptional vector bundle on $\tilde{X}$.*

Let $\tilde{\mathcal{E}}$ denote the vector bundle on $\tilde{\mathcal{X}}$ obtained by deforming $\tilde{E}$. Let $\mathcal{E} := (\pi_*\tilde{\mathcal{E}})^{\vee\vee}$ be the reflexive hull of the pushforward of $\tilde{\mathcal{E}}$ to $\mathcal{X}'$. Then $\mathcal{E}|_{\mathcal{X}'_t}$ is an exceptional vector bundle on $\mathcal{X}'_t$ for $t \neq 0$ and $E := \mathcal{E}|_X$ is a torsion-free sheaf on X.

Let $A := (p_*\tilde{A})^{\vee\vee}$ be the reflexive hull of the pushforward of $\tilde{A}$ to X. Then the reflexive hull $E^{\vee\vee}$ of E equals $A^{\oplus n}$ and $E^{\vee\vee}/E$ is a torsion sheaf supported at $P \in X$.

Remark 4.3. We show in Proposition 5.1 that such bundles G exist by induction on a. The induction step uses Proposition 4.2.

Proof. Since $\tilde{X} = \tilde{X}_1 \cup \tilde{X}_2$ has orbifold normal crossing singularities, the curve C is the scheme theoretic intersection of $\tilde{X}_1$ and $\tilde{X}_2$. So, fixing identifications $\tilde{A}^{\oplus n}|_C \simeq \mathcal{O}_C(1)^{\oplus n} \simeq G|_C$, we can construct a vector bundle $\tilde{E}$ on $\tilde{X}$ such that $\tilde{E}|_{\tilde{X}_1} = \tilde{A}^{\oplus n}$ and $\tilde{E}|_{\tilde{X}_2} = G$, see Lemma 7.3. (Note that the isomorphism type of $\tilde{E}$ does not depend on the choice of the identifications because $\mathrm{Aut}(\tilde{A}^{\oplus n}) = \mathrm{Aut}(\mathcal{O}_C(1)^{\oplus n}) = \mathrm{GL}(n, \mathbb{C})$.)

We show that $\tilde{E}$ is exceptional. Consider the exact sequence

$$0 \to \mathcal{E}nd\, \tilde{E} \to \mathcal{E}nd\, \tilde{E}|_{\tilde{X}_1} \oplus \mathcal{E}nd\, \tilde{E}|_{\tilde{X}_2} \to \mathcal{E}nd\, \tilde{E}|_C \to 0,$$

which is equal to

$$0 \to \mathcal{E}nd\, \tilde{E} \to \mathcal{O}_{\tilde{X}_1}^{n\times n} \oplus \mathcal{E}nd\, G \to \mathcal{O}_C^{n\times n} \to 0.$$

The curve C is smooth and rational, so $H^1(\mathcal{O}_C) = 0$, and $H^i(\mathcal{O}_{\tilde{X}_1}) = 0$ for $i > 0$ by Lemma 4.1. We deduce that $H^i(\mathcal{E}nd\, \tilde{E}) = H^i(\mathcal{E}nd\, G)$ for all i. Hence $\tilde{E}$ is exceptional because G is exceptional.

The vector bundle $\tilde{E}$ deforms uniquely to a vector bundle $\tilde{\mathcal{E}}$ over $\tilde{\mathcal{X}}$ because $\tilde{E}$ is exceptional. (Indeed, let $M/(0 \in T)$ denote the moduli space of simple sheaves on $\tilde{\mathcal{X}}/(0 \in T)$ [AK80, 7.4]. Then the vanishing $\mathrm{Ext}^1(\tilde{E}, \tilde{E}) = \mathrm{Ext}^2(\tilde{E}, \tilde{E}) = 0$ implies that $M/(0 \in T)$ is smooth of relative dimension 0 at $[\tilde{E}] \in M_0$ (see e.g. [H10, 7.1]).) The restrictions of $\tilde{\mathcal{E}}$ to the fibers of $\tilde{\mathcal{X}}/T$ are exceptional by upper semicontinuity of cohomology. (For t in some neighborhood of $0 \in T$ we have $H^0(\mathcal{E}nd\, \tilde{\mathcal{E}}_t) = \mathbb{C}$ and $\dim \mathrm{Ext}^i(\tilde{\mathcal{E}}_t, \tilde{\mathcal{E}}_t) = \dim H^i(\mathcal{E}nd\, \tilde{\mathcal{E}}_t) = 0$ for $i > 0$ by [H77, III.12.8]).)

Note that $E := \mathcal{E}|_X$ is torsion-free because $\mathcal{E}$ is reflexive. Indeed, $\mathcal{E}$ satisfies Serre's condition S_2 and $X = (t = 0) \subset \mathcal{X}$ is a Cartier divisor. So the restriction $E = \mathcal{E}|_X$ satisfies S_1, that is, E is torsion-free.

By construction $E|_{X\setminus\{P\}} = A^{\oplus n}|_{X\setminus\{P\}}$, so $E^{\vee\vee} = A^{\oplus n}$ and $E^{\vee\vee}/E$ is supported at P. □

Proposition 4.4. *Suppose $\mathcal{H}$ is a line bundle on $\mathcal{X}/T$ which is ample on fibers. Then for $t \neq 0$ the exceptional vector bundle $\mathcal{E}|_{\mathcal{X}'_t}$ on $\mathcal{X}'_t$ constructed in Proposition 4.2 is slope stable with respect to $\mathcal{H}|_{\mathcal{X}'_t}$.*

Proof. The exceptional divisor $W = \tilde{X}_2$ of π is $\mathbb{Q}$-Cartier by Proposition 3.1. Fix $M \in \mathbb{N}$ such that MW is Cartier. Let $\mathcal{H}'$ be the pullback of $\mathcal{H}$ to $\mathcal{X}'$, and define

$$\tilde{\mathcal{H}} := \pi^*\mathcal{H}'^{\otimes N} \otimes \mathcal{O}_{\tilde{\mathcal{X}}}(-MW)$$

for $N \gg 0$, Then $\tilde{\mathcal{H}}$ is a line bundle on $\tilde{\mathcal{X}}/T'$ which is ample on fibers, and its restriction to $\mathcal{X}'_t$ for $t \neq 0$ coincides with the restriction of $\mathcal{H}^{\otimes N}$. (Indeed, since $\tilde{X}$ has normal crossing singularities $(xy = 0) \subset \mathbb{A}^3_{x,y,z}$ generically along C, the one parameter deformation $\tilde{\mathcal{X}}$ has singularities of type

$$(xy = u \cdot t^l) \subset \mathbb{A}^3_{x,y,z} \times \mathbb{A}^1_t$$

generically along C, where u is a unit and $l \in \mathbb{N}$. Thus $-W|_W = X'|_W = \frac{1}{l}C$. So $-W$ is π-ample on $\tilde{\mathcal{X}}$ because $C \in |\mathcal{O}_W(n)|$ is ample on W. The line bundle $\mathcal{H}'$ is relatively ample on $\mathcal{X}'/T'$ by assumption. It follows that $\tilde{\mathcal{H}}$ is relatively ample on $\tilde{\mathcal{X}}/T'$ (see e.g. [KM98, 1.45]).) In what follows we write $\mu(F)$ for the slope of a sheaf F on a surface $S \subset \tilde{\mathcal{X}}$ defined using the polarization on S given by the restriction of $\tilde{\mathcal{H}}$, that is,

$$\mu(F) = \deg F/\operatorname{rk} F := (c_1(F) \cdot \tilde{\mathcal{H}}|_S)/\operatorname{rk}(F)$$

Suppose $\mathcal{E}|_{\mathcal{X}'_t}$ is not slope stable with respect to $\mathcal{H}|_{\mathcal{X}'_t}$ for $t \neq 0$. Then, by the argument for openness of stability [HL97, 2.3.1], after a finite surjective base change (which we suppress in our notation), there is a coherent sheaf $\mathcal{R}$ on $\mathcal{X}'$ and a surjection $\tilde{\mathcal{E}} \to \mathcal{R}$ such that $\mathcal{R}$ is flat over T', $0 < \operatorname{rk}(\mathcal{R}) < \operatorname{rk}(\tilde{\mathcal{E}})$, and $\mu(\mathcal{R}|_{\mathcal{X}'_t}) \leq \mu(\tilde{\mathcal{E}}|_{\mathcal{X}'_t})$ for all $t \in T'$. (Note that, by flatness of $\mathcal{R}$ over T', $\mathcal{R}|_{\tilde{X}}$ is a sheaf of constant rank on the reducible surface $\tilde{X}$. Thus $\mu(\mathcal{R}|_{\tilde{X}})$ is well-defined.)

Let $\tilde{E} \to R$ and $\tilde{E}_i \to R_i$ denote the restrictions of $\tilde{\mathcal{E}} \to \mathcal{R}$ to $\tilde{X}$ and $\tilde{X}_i$ for $i = 1, 2$. Recall that $\tilde{E}_1$ is the direct sum of n copies of the line bundle $\tilde{A}$ and $\tilde{E}_2$ is the exceptional vector bundle G on W. In particular $\tilde{E}_1$ is slope semistable, and $\tilde{E}_2$ is slope stable by Proposition 5.6 below. Thus $\mu(R_1) \geq \mu(\tilde{E}_1)$ and $\mu(R_2) > \mu(\tilde{E}_2)$. We deduce that $\mu(R) = \sum \mu(R_i) > \mu(\tilde{E}) = \sum \mu(\tilde{E}_i)$, a contradiction. □

5. Localized exceptional bundles

Proposition 5.1. *Let n, a be positive integers such that $a < n$ and $(a, n) = 1$. Write*

$$W = W_{n,a} := (XY = Z^n + T^a) \subset \mathbb{P}(1, na-1, a, n)$$

Let C_1 and C_2 be the smooth rational curves on W defined by $C_1 = (Z = 0)$ and $C_2 = (T = 0)$. Then there exist exceptional vector bundles F_1 and F_2 on W of ranks a and n such that for each $j = 1, 2$ we have

1. *$H^i(F_j^\vee) = 0$ for all i,*
2. *$H^i(F_j) = 0$ for $i > 0$,*
3. *F_j is generated by global sections, and*
4. *$F_j|_{C_j} \simeq \mathcal{O}_{C_j}(1)^{\oplus \operatorname{rk} F_j}$.*

(Here $\mathcal{O}_{C_j}(1)$ denotes the line bundle of degree 1 on the smooth rational curve C_j.)

Construction 5.2. The proof of Proposition 5.1 uses the following degeneration of W. Consider the one parameter family of normal surfaces

$$\mathcal{X} = (XY = tZ^n + T^a) \subset \mathbb{P}(1, na-1, a, n) \times \mathbb{A}^1_t.$$

Note that $\mathcal{X}_t \simeq W$ for $t \neq 0$. The special fiber $X := \mathcal{X}_0$ is isomorphic to the weighted projective plane $\mathbb{P}(1, na-1, a^2)$ via the morphism

$$\mathbb{P}(1, na-1, a^2) \to X = (XY = T^a) \subset \mathbb{P}(1, na-1, a, n),$$

$$(U, V, W) \mapsto (X, Y, Z, T) = (U^a, V^a, W, UV).$$

The surface X has two singular points $P = (0\colon 0\colon 1)$ and $Q = (0\colon 1\colon 0)$. The point P is a Wahl singularity of type $\frac{1}{a^2}(1, ab-1)$ where $b = n \bmod a$, and the germ $(P \in \mathcal{X})/(0 \in \mathbb{A}^1_t)$ is a $\mathbb{Q}$-Gorenstein smoothing of $(P \in X)$. The point Q is a cyclic quotient singularity of type $\frac{1}{na-1}(1, a^2)$ and the deformation $\mathcal{X}/(0 \in \mathbb{A}^1_t)$ is locally trivial near Q.

Proof of Proposition 5.1. We first construct F_2 given F_1. The vector bundle F_1 is globally generated, that is, the natural morphism

$$H^0(F_1) \otimes \mathcal{O}_W \to F_1 \tag{5.1}$$

is surjective. We define F_2 as the dual of the kernel of (5.1). Thus F_2 is a vector bundle such that $\mathrm{rk}(F_2) = h^0(F_1) - \mathrm{rk}(F_1)$ and $c_1(F_2) = c_1(F_1)$. The bundle F_2 is exceptional and $H^i(F_2^\vee) = 0$ for all i by [G90, §2.4]. Indeed, in the terminology of op. cit., the pair $(\mathcal{O}_W, F_1)$ is an *exceptional pair*, and $(F_2^\vee, \mathcal{O}_W)$ is its *left mutation*. Moreover, the exact sequence

$$0 \to F_1^\vee \to H^0(F_1)^* \otimes \mathcal{O}_W \to F_2 \to 0$$

shows that F_2 is globally generated and $H^i(F_2) = 0$ for $i > 0$.

The surface W has only quotient singularities and satisfies $H^1(\mathcal{O}_W) = H^2(\mathcal{O}_W) = 0$, so the class group $\mathrm{Cl}(W)$ is identified with $H_2(W, \mathbb{Z})$, see §7.1. By Lemma 5.4 the homology group $H_2(W, \mathbb{Z})$ is isomorphic to $\mathbb{Z}$, generated by the restriction h of the positive generator $h_{\mathbb{P}} = c_1(\mathcal{O}_{\mathbb{P}}(1))$ of $H_4(\mathbb{P}, \mathbb{Z})$, where $\mathbb{P} = \mathbb{P}(1, na-1, a, n)$ denotes the ambient weighted projective space. (See §7.2.1 for background on weighted projective spaces.) Note that $h^2 = 1/(na-1)$ (because $h^2 = h_{\mathbb{P}}^2 \cdot W = (na)h_{\mathbb{P}}^3$ and $h_{\mathbb{P}}^3 = 1/((na-1)an)$). Let $H = (X = 0) \subset W$, an effective Weil divisor with class h. Now $C_1 = (Z = 0) \sim aH$ and $F_1|_{C_1} \simeq \mathcal{O}_{C_1}(1)^{\oplus a}$, thus $c_1(F_1) \cdot ah = a$ and so $c_1(F_1) = (na-1)h$. Since F_1 is an exceptional bundle of rank a we have

$$c_2(F_1) = \frac{a-1}{2a}(c_1(F_1)^2 + a + 1),$$

see Lemma 5.3 below. The canonical class K_W of W is given by the adjunction formula

$$K_W = (K_{\mathbb{P}} + W)|_W \sim -(a+n)H.$$

Now the Riemann–Roch formula (see Lemma 7.1) gives

$$h^0(F_1) = \chi(F_1) = a\chi(\mathcal{O}_W) + \frac{1}{2}c_1(F_1)(c_1(F_1) - K_W) - c_2(F_1) = a + n.$$

Thus F_2 is a vector bundle of rank n.

It remains to show that $F_2|_{C_2} \simeq \mathcal{O}_{C_2}(1)^{\oplus n}$. The bundle $F_2|_{C_2}$ on $C_2 \simeq \mathbb{P}^1$ has rank n and degree

$$c_1(F_2) \cdot C_2 = c_1(F_1) \cdot nh = (na-1)nh^2 = n.$$

So it suffices to show that $F_2|_{C_2}$ is rigid, that is, $H^1(\mathcal{E}nd\, F_2|_{C_2}) = 0$. Consider the exact sequence

$$0 \to \mathcal{E}nd\, F_2(-C_2) \to \mathcal{E}nd\, F_2 \to \mathcal{E}nd\, F_2|_{C_2} \to 0.$$

We have $H^1(\mathcal{E}nd\, F_2) = 0$ and

$$H^2(\mathcal{E}nd\, F_2(-C_2)) = H^0(\mathcal{E}nd\, F_2(K_W + C_2))^* = H^0(\mathcal{E}nd\, F_2(-aH))^* = 0$$

using Serre duality (see §7.1), $K_W + C_2 \sim -aH < 0$, and $\operatorname{End} F_2 = \mathbb{C}$. So $H^1(\mathcal{E}nd\, F_2|_{C_2}) = 0$ as required.

We now prove the existence of F_1 by induction on a. If $a = 1$ then $W = \mathbb{P}(1, n-1, 1)$ and we can take $F_1 = \mathcal{O}_W(n-1)$. (Indeed, the vanishings (1) and (2) follow from the description of the cohomology of the sheaves $\mathcal{O}(n)$ on weighted projective spaces [D82, 1.4.1]. The line bundle $\mathcal{O}_W(n-1)$ is globally generated by X^{n-1}, Y, Z^{n-1}. The restriction $\mathcal{O}_W(n-1)|_C$ has degree $(n-1)h \cdot h = 1$, thus $\mathcal{O}_W(n-1)|_C \simeq \mathcal{O}_C(1)$.)

Now suppose $a > 1$. Consider the degeneration $\mathcal{X}/(0 \in T)$ of W described in Construction 5.2. Write $n = ka+b$, $0 < b < a$. The point $P = (0\colon 0\colon 1) \in X = \mathbb{P}(1, na-1, a^2)$ is a Wahl singularity of type $\frac{1}{a^2}(1, ab-1)$, and the germ $(P \in \mathcal{X})/(0 \in T)$ is a $\mathbb{Q}$-Gorenstein smoothing of $(P \in X)$. Let $(0 \in T') \to (0 \in T)$ and $\pi\colon \tilde{\mathcal{X}} \to \mathcal{X}'$ be the base change and blowup of Proposition 3.1. The exceptional divisor of π is $\tilde{X}_2 = W_{a,b}$. By induction and the construction of F_2 from F_1 above, there exists an exceptional vector bundle G on $\tilde{X}_2$ of rank a such that $H^i(G^\vee) = 0$ for all i, $H^i(G) = 0$ for $i > 0$, G is globally generated, and $G|_C \simeq \mathcal{O}_C(1)^{\oplus a}$, where $C = \tilde{X}_1 \cap \tilde{X}_2$ is the double curve of $\tilde{X} = \tilde{\mathcal{X}}_0$.

The surface $X = \mathbb{P}(1, na-1, a^2)$ is the toric variety associated to a free abelian group N and a fan Σ in $N \otimes_{\mathbb{Z}} \mathbb{R}$ as follows. The group N has rank 2 and is generated by vectors v_0, v_1, v_2 satisfying the relation

$$v_0 + (na-1)v_1 + a^2 v_2 = 0.$$

The fan Σ is the complete fan with rays generated by v_0, v_1, v_2. The birational morphism $p\colon \tilde{X}_1 \to X$ is the weighted blowup of the point $P = (0\colon 0\colon 1) \in X$ with weights $\frac{1}{a^2}(1, ab-1)$ with respect to the orbifold coordinates $u = U/W^{1/a^2}, v = V/W^{(na-1)/a^2}$. The morphism p corresponds to the refinement $\tilde{\Sigma}$ of the fan Σ obtained by adding the ray generated by

$$w := \tfrac{1}{a^2}(v_0 + (ab-1)v_1) \in N.$$

Let D be the divisor $(V = 0) \subset X = \mathbb{P}(1, na-1, a^2)$ and $D' \subset \tilde{X}_1$ its strict transform. Note that $D' \subset \tilde{X}_1$ is the toric boundary divisor corresponding to the ray $\mathbb{R}_{\geq 0} \cdot v_1$ of $\tilde{\Sigma}$. The divisor D' is Cartier and $D' \cdot C = 1$ by Proposition 3.1(5). By Proposition 4.2 there is an exceptional vector bundle

$\tilde{E}$ on $\tilde{X}$ obtained by glueing $\mathcal{O}_{\tilde{X}_1}(D')^{\oplus a}$ on $\tilde{X}_1$ and G on $\tilde{X}_2$ along $\mathcal{O}_C(1)^{\oplus a}$ on C, and $\tilde{E}$ deforms to an exceptional bundle F_1 on the general fiber W of $\tilde{\mathcal{X}}/T$. It remains to show that F_1 satisfies the properties (1)–(4) in the statement.

(1,2) It suffices to verify the corresponding vanishings for $\tilde{E}$. We have the exact sequence of sheaves on $\tilde{X}$

$$0 \to \tilde{E}^\vee \to \mathcal{O}_{\tilde{X}_1}(-D')^{\oplus a} \oplus G^\vee \to \mathcal{O}_C(-1)^{\oplus a} \to 0.$$

Now $H^i(G^\vee) = 0$ for all i by assumption, $H^i(\mathcal{O}_C(-1)) = 0$ for all i, and we find $H^i(\mathcal{O}_{\tilde{X}_1}(-D')) = 0$ for all i using the exact sequence

$$0 \to \mathcal{O}_{\tilde{X}_1}(-D') \to \mathcal{O}_{\tilde{X}_1} \to \mathcal{O}_{D'} \to 0.$$

(Note that the toric boundary divisor D' is a smooth rational curve.) Hence $H^i(\tilde{E}^\vee) = 0$ for all i. Similarly, for (2) we consider the exact sequence

$$0 \to \tilde{E} \to \mathcal{O}_{\tilde{X}_1}(D')^{\oplus a} \oplus G \to \mathcal{O}_C(1)^{\oplus a} \to 0. \tag{5.2}$$

A toric calculation shows that $(D')^2 = k > 0$, where $n = ka + b$ as above. (Indeed, with notation as above, we compute $v_2 + w = -kv_1$, so $(D')^2 = k$ by [F93], p. 44.) The exact sequence

$$0 \to \mathcal{O}_{\tilde{X}_1} \to \mathcal{O}_{\tilde{X}_1}(D') \to \mathcal{O}_{D'}(k) \to 0$$

shows $H^i(\mathcal{O}_{\tilde{X}_1}(D')) = 0$ for $i > 0$. The exact sequences

$$0 \to \mathcal{O}_{\tilde{X}_1}(-C) \to \mathcal{O}_{\tilde{X}_1}(D' - C) \to \mathcal{O}_{D'}(k-1) \to 0$$

and

$$0 \to \mathcal{O}_{\tilde{X}_1}(-C) \to \mathcal{O}_{\tilde{X}_1} \to \mathcal{O}_C \to 0$$

show that $H^1(\mathcal{O}_{\tilde{X}_1}(D'-C)) = 0$, hence the restriction map $H^0(\mathcal{O}_{\tilde{X}_1}(D')) \to H^0(\mathcal{O}_C(1))$ is surjective. We deduce that $H^i(\tilde{E}) = 0$ for $i > 0$.

(3) Global sections of $\tilde{E}$ lift to global sections of $\tilde{\mathcal{E}}$ because $H^1(\tilde{E}) = 0$ [H77, III.12.11]. So it suffices to prove that $\tilde{E}$ is globally generated. The bundle G is globally generated by assumption and the restriction map $H^0(\mathcal{O}_{\tilde{X}_1}(D')) \to H^0(\mathcal{O}_C(1))$ is surjective as proved above. It follows that $\tilde{E}$ is globally generated along $\tilde{X}_2$. (Here, given a coherent sheaf F on a scheme X and a closed subscheme $Y \subset X$, we say F is *globally generated along* Y if F is generated by $H^0(X, F)$ along Y.)

Let S denote the support of the cokernel of the natural map

$$H^0(\tilde{E}) \otimes \mathcal{O}_{\tilde{X}_1} \to \tilde{E}|_{\tilde{X}_1}.$$

Thus $\tilde{E}$ is globally generated iff $S = \emptyset$. We first show that S is a union of toric strata of the toric surface $\tilde{X}_1$. The action σ of the big torus $H \simeq (\mathbb{C}^*)^2$ on $\tilde{X}_1$ restricts to an action on C given by

$$C = (T = 0) \subset \tilde{X}_2 = (XY = Z^a + T^b) \subset \mathbb{P}(1, ab-1, b, a)$$
$$H \ni h\colon (X, Y, Z) \mapsto (X, \chi(h)^a Y, \chi(h) Z)$$

for some character χ of H. Consider the action σ' of H on $\tilde{X}_1$ given by $\sigma'(h,x) = \sigma(h^b, x)$. Then σ' extends to an action of H on $\tilde{X}$ defined on $\tilde{X}_2$ by

$$h\colon (X,Y,Z,T) \mapsto (X, \chi(h)^{ab}Y, \chi(h)^b Z, \chi(h)^a T).$$

Since $\tilde{E}$ is exceptional, it has no nontrivial deformations, so $h^*\tilde{E}$ is isomorphic to $\tilde{E}$ for all $h \in H$. It follows that $S \subset \tilde{X}_1$ is a union of toric strata. Now $\tilde{E}$ is globally generated along C, so it suffices to show that $\tilde{E}$ is globally generated along the toric boundary divisor $B \subset \tilde{X}_1$ disjoint from C. Note that B maps isomorphically to $(W=0) \subset X$ under p.

The divisor D' intersects each of B and C transversely in a smooth point of $\tilde{X}_1$. In particular, $\tilde{E}|_B \simeq \mathcal{O}_B(1)^{\oplus a}$. Let $f\colon \tilde{X}_1' \to \tilde{X}_1$ be the blowup of k distinct interior points of D' so that the strict transform D'' satisfies $D'' = f^*D' - \sum_{i=1}^k E_i$ where the E_i are the exceptional curves. Then $(D'')^2 = 0$ and so D'' defines a ruling of $\tilde{X}_1'$ with sections B and C. So $H^0(\mathcal{O}_{\tilde{X}_1'}(D'')) \subset H^0(\mathcal{O}_{\tilde{X}_1}(D'))$ maps isomorphically to $H^0(\mathcal{O}_B(1))$ and $H^0(\mathcal{O}_C(1))$. Since $G = \tilde{E}|_{\tilde{X}_2}$ is globally generated along C we deduce that $\tilde{E}$ is globally generated along B.

(4) The divisor $\mathcal{B} := (Z=0) \subset \mathcal{X}/(0 \in T)$ is a $\mathbb{P}^1$-bundle over the base with fiber $C_2 \subset W = \mathcal{X}_t$ for $t \neq 0$ and $B = (W=0) \subset X$ for $t=0$. As noted above we have $\tilde{E}|_B \simeq \mathcal{O}_B(1)^{\oplus a}$, so also $F_1|_{C_2} \simeq \mathcal{O}_{C_2}(1)^{\oplus a}$. □

Lemma 5.3. *Let W be a projective normal surface with only quotient singularities and F an exceptional vector bundle of rank n on W. Then*

$$c_2(F) = \frac{n-1}{2n}(c_1(F)^2 + n + 1).$$

Proof. By the Riemann–Roch formula for $\mathcal{E}nd(F)$ on W (see Lemma 7.1) we have

$$\chi(\mathcal{E}nd\, F) = n^2\chi(\mathcal{O}_W) + (n-1)c_1(F)^2 - 2nc_2(F). \tag{5.3}$$

The locally free sheaf F is exceptional, that is, $H^0(\mathcal{E}nd\, F) = \mathbb{C}$ and $H^i(\mathcal{E}nd\, F) = 0$ for $i > 0$. The sheaf $\mathcal{O}_W$ is a direct summand of $\mathcal{E}nd\, F$, so we also have $H^i(\mathcal{O}_W) = 0$ for $i > 0$. Thus $\chi(\mathcal{E}nd\, F) = \chi(\mathcal{O}_W) = 1$. Now solving (5.3) for $c_2(F)$ yields the formula in the statement. □

Lemma 5.4. *Let $W = (XY = Z^n + T^a) \subset \mathbb{P}(1, na-1, a, n)$. Then $H_2(W,\mathbb{Z})$ is isomorphic to $\mathbb{Z}$, generated by the restriction of the positive generator $h := c_1(\mathcal{O}_{\mathbb{P}}(1)) \in H_4(\mathbb{P},\mathbb{Z})$ of the homology of the ambient weighted projective space $\mathbb{P} = \mathbb{P}(1, na-1, a, n)$.*

Proof. By Construction 5.2 the surface W is obtained from the weighted projective plane $X = \mathbb{P}(1, na-1, a^2) \subset \mathbb{P}$ by smoothing the Wahl singularity $P \in X$ of type $\frac{1}{a^2}(1, ab-1)$. We have $h|_X = al$, where l denotes the positive generator of $H_2(X,\mathbb{Z})$. Now by Lemma 5.5 below we deduce that $H_2(W,\mathbb{Z})$ is generated by $h|_W$ as required. □

Lemma 5.5. *Let X be a projective normal surface and $P \in X$ a Wahl singularity of type $\frac{1}{n^2}(1, na-1)$. Let L denote the link of $P \in X$. Assume that $H_2(X, \mathbb{Z}) \to H_1(L, \mathbb{Z})$ is surjective. Let $\mathcal{X}/(0 \in T)$ be a deformation of X over a smooth curve germ such that the germ $(P \in \mathcal{X})/(0 \in T)$ is a $\mathbb{Q}$-Gorenstein smoothing of $(P \in X)$ and the deformation $\mathcal{X}/T$ is locally trivial elsewhere. Let Y denote a general fiber of $\mathcal{X}/T$. Then the specialization map*

$$\mathrm{sp}\colon H_2(Y, \mathbb{Z}) \to H_2(X, \mathbb{Z})$$

is injective with cokernel isomorphic to $\mathbb{Z}/n\mathbb{Z}$.

Proof. Let $B \subset X$ be the intersection of X with a small closed ball about P in some embedding, and X^o the complement of the interior of B. Then B is contractible and has boundary L, the link of $P \in X$. Let $M \subset Y$ be the Milnor fiber of the smoothing of $P \in X$. Then the Mayer–Vietoris sequences for $Y = X^o \cup M$ and $X = X^o \cup B$ give a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_2(X^o) & \longrightarrow & H_2(Y) & \longrightarrow & H_1(L) & \longrightarrow & H_1(X^o) \oplus H_1(M) \\ & & \| & & \downarrow & & \| & & \downarrow \\ 0 & \longrightarrow & H_2(X^o) & \longrightarrow & H_2(X) & \longrightarrow & H_1(L) & \longrightarrow & H_1(X^o) \end{array}$$

using $H_2(L) = H_2(M) = 0$ and contractibility of B. Now $H_2(X) \to H_1(L)$ is surjective by assumption and $H_1(L) \to H_1(M)$ is a surjection of the form $\mathbb{Z}/n^2\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. It follows that $H_2(Y) \to H_2(X)$ is injective with cokernel $H_1(M)$ isomorphic to $\mathbb{Z}/n\mathbb{Z}$ as claimed. $\square$

Proposition 5.6. *Let $W = (XY = Z^n + T^a) \subset \mathbb{P}(1, na-1, a, n)$ and $C = (T = 0) \subset W$, a smooth rational curve. Let G be an exceptional vector bundle on W such that $G|_C$ is semistable. Then G is slope stable.*

Proof. Let A be an ample line bundle on W. For a coherent sheaf F on W we define the slope $\mu(F) = (c_1(F) \cdot A)/\operatorname{rk}(F)$. (Note that the choice of A is irrelevant because $b_2(W) = 1$, see Lemma 5.4.) For a coherent sheaf F on an irreducible projective curve Γ, we define the slope $\mu(F) = \deg(F)/\operatorname{rk}(F)$.

Suppose for a contradiction that G is not slope stable. So there is a surjection $G \to R$ where R is a torsion-free sheaf such that $\mu(R) \leq \mu(G)$ and $\operatorname{rk}(R) < \operatorname{rk}(G)$. Let $S := R^{\vee\vee}$ be the reflexive hull of R. We first show that S is locally free. Consider the smooth rational curve $C = (T = 0) \subset W$. Write $[C] = \alpha c_1(A)$, some $\alpha \in \mathbb{Q}$, $\alpha > 0$. Then $\mu(G|_C) = \alpha\mu(G)$. For F a coherent sheaf on W and $\Gamma \subset W$ an irreducible curve, let $\bar{F}_\Gamma$ denote the quotient of $F|_\Gamma$ by its torsion subsheaf. We have a surjection $G|_C \to \bar{R}_C$, and an injection $\bar{R}_C \to \bar{S}_C$ with torsion cokernel (because the inclusion $R \subset S$ is an isomorphism outside a finite set). In particular $\mu(\bar{R}_C) \leq \mu(\bar{S}_C)$. By Lemma 5.7 below,

$$\mu(\bar{S}_C) \leq (c_1(S) \cdot [C])/\operatorname{rk}(S) = \alpha\mu(S),$$

with equality iff S is locally free. Combining, we find

$$\mu(\bar{R}_C) \le \mu(\bar{S}_C) \le \alpha\mu(S) = \alpha\mu(R) \le \alpha\mu(G) = \mu(G|_C)$$

with equality only if S is locally free. But $\mu(G|_C) \le \mu(\bar{R}_C)$ by semistability of $G|_C$, so S is locally free as claimed.

Let $D \in |-K_W|$ be a general member of the anticanonical linear system on W. Then D is an irreducible curve of arithmetic genus 1. (Indeed, D is irreducible and reduced by explicit calculation, and we have

$$2p_a(D) - 2 = (K_W + D) \cdot D = 0$$

by the adjunction formula (which is valid for the irreducible curve D on the singular surface W because $K_W + D$ is Cartier [K92, 16.4.3]).) We show that $G|_D$ is slope stable. Consider the exact sequence

$$0 \to \mathcal{E}nd\, G(-D) \to \mathcal{E}nd\, G \to \mathcal{E}nd\, G|_D \to 0.$$

We have $H^i(\mathcal{E}nd\, G) = 0$ for $i \neq 0$ because G is exceptional, and

$$H^i(\mathcal{E}nd\, G(-D)) = H^i(\mathcal{E}nd\, G(K_W)) = H^{2-i}(\mathcal{E}nd\, G)^* = 0$$

for $i \neq 2$ by Serre duality. Thus $H^0(\mathcal{E}nd\, G|_D) = H^0(\mathcal{E}nd\, G) = \mathbb{C}$, that is, $G|_D$ is simple. Now by [BK06, 4.13] $G|_D$ is slope stable.

Finally, consider the surjection $G|_D \to \bar{R}_D$. Recall that the reflexive hull S of R is locally free. Write $[D] = \beta c_1(A)$, where $\beta \in \mathbb{Q}$, $\beta > 0$. Then

$$\mu(\bar{R}_D) \le \mu(S|_D) = \beta\mu(S) = \beta\mu(R) \le \beta\mu(G) = \mu(G|_D).$$

This contradicts the stability of $G|_D$. □

Lemma 5.7. *Let F be a reflexive sheaf on W. Let $\bar{F}_C$ denote the quotient of the restriction $F|_C$ by its torsion subsheaf. Then*

$$\deg(\bar{F}_C) \le c_1(F) \cdot [C],$$

with equality iff F is locally free.

Proof. Recall that the surface W has a unique singular point

$$(Q \in W) \simeq (0 \in \mathbb{C}^2_{z,t}/\tfrac{1}{na-1}(a,n)),$$

a cyclic quotient singularity with group of order $r := na - 1$. Moreover the curve C is a smooth rational curve passing through Q, étale locally at Q given by $(t = 0) \subset (Q \in W)$. The sheaf F is locally free away from Q (because a reflexive sheaf on a smooth surface is locally free, see e.g. [H80, 1.4]). Let $f\colon (P \in V) \to (Q \in U)$ be a local smooth Galois cover of an étale neighbourhood U of $Q \in W$ with group $G = \mathbb{Z}/r\mathbb{Z}$. Let $F_V = (f^*F)^{\vee\vee}$ be the reflexive hull of the pullback of F. Then F_V is locally free and $F|_U = (f_*F_V)^G$. (Indeed $F|_U$ and $(f_*F_V)^G$ coincide away from Q (where f is étale), so it suffices to show that $(f_*F_V)^G$ is reflexive. The sheaf f_*F_V is the pushforward of a reflexive sheaf by a finite surjective morphism, so is reflexive [H80, 1.7]. The subsheaf $(f_*F_V)^G \subset f_*F_V$ of G-invariant sections is a direct summand, hence also reflexive.) We trivialize F_V and diagonalize the G-action to obtain

$$F_V \simeq \oplus_{i=0}^{r-1} \mathcal{O}_V^{\oplus m_i}$$

where the $m_i \in \mathbb{N} \cup \{0\}$ and the generator $g = 1 \in G = \mathbb{Z}/r\mathbb{Z}$ acts on the components of the ith summand by

$$g \cdot f = g(f) \cdot \zeta^{-i}, \quad \zeta = \exp(2\pi i/r).$$

Let N be an étale neighbourhood of C in W such that there exists an effective Weil divisor $D \subset N$ given étale locally at Q by $(z = 0) \subset (Q \in W)$ and otherwise disjoint from C. Then, replacing U by its restriction to N and recalling that $g(z) = \zeta z$, we find

$$F|_U \simeq \oplus_{i=0}^{r-1} \mathcal{O}_U(-iD)^{\oplus m_i}.$$

Thus, replacing N by $(N \setminus D) \cup U$, there is a locally free sheaf F' on N and a short exact sequence

$$0 \to F|_N \to F' \to \oplus_{i=1}^{r-1} \mathcal{O}_{iD}^{\oplus m_i} \to 0.$$

In particular, since $D \cdot C = \frac{1}{r}$,

$$c_1(F) \cdot [C] = c_1(F') \cdot [C] - \sum_{i=1}^{r-1} m_i \cdot \frac{i}{r}.$$

For each $0 < i < r$ the cokernel of the map $\mathcal{O}_N(-iD)|_C \to \mathcal{O}_C$ is the skyscraper sheaf $k(Q)$ at Q with stalk $\mathbb{C}$ (by direct computation). Thus the cokernel of the map $F|_C \to F'|_C$ is isomorphic to $\oplus_{i=1}^{r-1} k(Q)^{\oplus m_i}$, and so

$$\deg(\bar{F}_C) = \deg(F'|_C) - \sum_{i=1}^{r-1} m_i.$$

We have $c_1(F') \cdot [C] = \deg(F'|_C)$ because F' is locally free. Combining, we find

$$\deg(\bar{F}_C) = c_1(F) \cdot [C] - \sum_{i=1}^{r-1} m_i \cdot \left(1 - \frac{i}{r}\right) \leq c_1(F) \cdot [C]$$

with equality iff $m_i = 0$ for $0 < i < r$, equivalently, F is locally free. □

Proof of Theorem 1.1. By Propositions 4.2 and 5.1, to establish the existence of the sheaf $\mathcal{E}$ satisfying 1.1(2)(a,b) it suffices to show that the hypotheses of Theorem 1.1 imply the hypothesis (1) of §4, namely, that the map

$$H_2(X, \mathbb{Z}) \to H_1(L, \mathbb{Z}) \tag{5.4}$$

is surjective, where L is the link of the singularity.

Let B denote the intersection of X with a small closed ball centered at the singularity $P \in X$ in some embedding, M the Milnor fiber of the smoothing, and $X^o \subset X$ the complement of the interior of B. As in the proof of Lemma 5.5, consider the Mayer-Vietoris sequences for $Y = X^o \cup M$

and $X = X^o \cup B$. Let I_X and I_Y denote the image of $H_2(X)$ and $H_2(Y)$ in $H_1(L)$. We obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_1(L)/I_Y & \longrightarrow & H_1(X^o) \oplus H_1(M) & \longrightarrow & H_1(Y) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & H_1(L)/I_X & \longrightarrow & H_1(X^o) & \longrightarrow & H_1(X) & \longrightarrow & 0 \end{array}$$

By the snake lemma we obtain an exact sequence

$$0 \to I_X/I_Y \to H_1(M) \to H_1(Y) \to H_1(X) \to 0.$$

Now $H_1(M) \simeq \mathbb{Z}/n\mathbb{Z}$ and $H_1(Y)$ is finite of order coprime to n by assumption. Hence $I_X/I_Y = H_1(M)$ and $H_1(Y) = H_1(X)$. Now consider the p-part of the exact sequence

$$H_1(L) \to H_1(X^o) \oplus H_1(M) \to H_1(Y) \to 0$$

for p a prime. If p divides n, then since $H_1(L) \simeq \mathbb{Z}/n^2\mathbb{Z}$, $H_1(M) \simeq \mathbb{Z}/n\mathbb{Z}$, and $H_1(Y)_{(p)} = 0$, we find that $H_1(X^o)_{(p)} = 0$. (Here for a $\mathbb{Z}$-module A and a prime $p \in \mathbb{N}$ we write $A_{(p)}$ for the localization of A at the prime ideal (p).) If p does not divide n, then $H_1(X^o)_{(p)} = H_1(Y)_{(p)}$. Thus $H_1(X^o) = H_1(Y)$. Combining, $H_1(X^o) = H_1(X)$. Now the Mayer–Vietoris sequence for $X = X^o \cup B$ shows that $H_2(X) \to H_1(L)$ is surjective as required.

The statement 1.1(1) holds by Lemma 5.5 and the surjectivity of (5.4) proved above. The stability statement is given by Proposition 4.4.

It remains to establish the stated properties of the Chern classes of F. We have $c_1(E) = nc_1(A) \in H_2(X, \mathbb{Z})$ because $E^{\vee\vee} = A^{\oplus n}$ and E is torsion-free. Moreover $c_1(F) = c_1(E) \in H_2(Y, \mathbb{Z}) \subset H_2(X, \mathbb{Z})$. (Indeed, working locally analytically over $(0 \in T)$, fix an embedding $(P \in \mathcal{X}) \subset ((0,0) \in \mathbb{C}^N \times T)$, let B be a small closed ball about 0 in $\mathbb{C}^N$, and define $\mathcal{X}^o = \mathcal{X} \setminus B \times T$. Then $\mathcal{X}^o/T$ is a fibration of smooth manifolds with boundary, with special fiber X^o and general fibre $Y^o \subset Y$ the complement of the interior of the Milnor fiber. The specialization map $H_2(Y, \mathbb{Z}) \to H_2(X, \mathbb{Z})$ is the composition

$$H_2(Y, \mathbb{Z}) = H^2(Y, \mathbb{Z}) \to H^2(Y^o, \mathbb{Z}) = H^2(X^o, \mathbb{Z}) = H_2(X, \mathbb{Z}),$$

where we have used Poincaré duality (see §7.1) and the identification of the cohomology of the fibers of $\mathcal{X}^o/T$. Since the class

$$c_1(\mathcal{E}|_{\mathcal{X}^o_t}) \in H^2(\mathcal{X}^o_t, \mathbb{Z}) = H^2(X^o, \mathbb{Z})$$

is independent of $t \in T$, it follows that $c_1(F)$ maps to $c_1(E)$ under the specialization map.) The formula for $c_2(F)$ holds because F is exceptional, see Lemma 5.3. Since $c_1(F) = nc_1(A)$ and $K_Y = K_X$ in $H_2(Y, \mathbb{Z}) \subset H_2(X, \mathbb{Z})$ we can compute $c_1(F) \cdot K_Y$ modulo n by a local computation at the singular point $P \in X$. Identify $(P \in X)$ with $(0 \in \mathbb{A}^2_{u,v}/\frac{1}{n^2}(1, na-1))$. By construction $c_1(A)$ is locally represented by the class of the Weil divisor $D := (v = 0)$ (because the strict transform D' of this divisor in $\tilde{X}_1$ is

Cartier and satisfies $D' \cdot C = 1$). The canonical divisor K_X is locally represented by $-(uv = 0) \sim -na(u = 0)$. Thus the local intersection number $(nc_1(A) \cdot K_X)_P$ is given by

$$(nc_1(A) \cdot K_X)_P = n(v = 0) \cdot (-na(u = 0)) = -n^2a/n^2 = -a \bmod n.$$

It follows that $c_1(A) = c_1(F)/n$ generates $H_2(X, \mathbb{Z})/H_2(Y, \mathbb{Z}) \simeq \mathbb{Z}/n\mathbb{Z}$ because $(a, n) = 1$. □

6. Example: The projective plane

We analyze our construction in the case $Y = \mathbb{P}^2$. We use the classification of exceptional bundles on Y [DLP85],[R89] and the classification of degenerations $Y \rightsquigarrow X$ [HP10] to establish a bijective correspondence, see Theorem 6.4.

Theorem 6.1. [HP10, 1.2] *Let X be a normal surface with quotient singularities which admits a smoothing to the projective plane. Then X is one of the following:*

(1) *A weighted projective plane $\mathbb{P}(a^2, b^2, c^2)$, where (a, b, c) is a solution of the Markov equation*

$$a^2 + b^2 + c^2 = 3abc. \tag{6.1}$$

(2) *A $\mathbb{Q}$-Gorenstein deformation of one of the toric surfaces in (1), determined by specifying a subset of the singularities to be smoothed.*

The solutions of the Markov equation are easily described as follows. The triple $(1, 1, 1)$ is a solution, and all solutions are obtained from $(1, 1, 1)$ by a sequence of *mutations* given by choosing one of the variables (c, say), holding the remaining variables fixed, and replacing the chosen variable by the other solution of the quadratic equation (6.1):

$$(a, b, c) \mapsto (a, b, c' = 3ab - c). \tag{6.2}$$

We can define a graph G with vertices labelled by solutions of the Markov equation and edges corresponding to pairs of solutions related by a single mutation. Then G is an infinite tree such that every vertex has degree 3. See [C57, II.3].

The surface $\mathbb{P} = \mathbb{P}(a^2, b^2, c^2)$ has cyclic quotient singularities of types $\frac{1}{a^2}(b^2, c^2)$, $\frac{1}{b^2}(c^2, a^2)$, $\frac{1}{c^2}(a^2, b^2)$. Using the Markov equation one sees that these are Wahl singularities (note that a, b, c are coprime and not divisible by 3 by the inductive description of the solutions of the Markov equation above). Moreover there are no locally trivial deformations and no local-to-global obstructions because $H^1(T_{\mathbb{P}}) = H^2(T_{\mathbb{P}}) = 0$. (Here for a variety X, possibly singular, we define the tangent sheaf $T_X := \mathcal{H}om(\Omega_X, \mathcal{O}_X)$ as the dual of the sheaf of Kähler differentials, cf. [H77, II.8].) Thus the versal $\mathbb{Q}$-Gorenstein deformation space of $\mathbb{P}$ maps isomorphically to the product of the versal $\mathbb{Q}$-Gorenstein deformation spaces of its singularities (see Lemma 7.2), which are smooth of dimension 1 (see §2).

Proposition 6.2. *Let X be a normal surface with quotient singularities which admits a smoothing to the projective plane. Then X is uniquely determined up to isomorphism by its singularities.*

Proof. If X is smooth then necessarily X is isomorphic to $\mathbb{P}^2$. Now suppose X has r singularities with indices $a_1, \ldots, a_r$. By Theorem 6.1 we have $r \leq 3$ and the surface X is obtained from a weighted projective plane $\mathbb{P} = \mathbb{P}(a_1^2, a_2^2, a_3^2)$ by smoothing the singularity of index a_i for each $i > r$, where (a_1, a_2, a_3) is a solution of the Markov equation. If $r = 3$ then $X = \mathbb{P}$ and clearly X is determined by its singularities. (Indeed, if $w_0, w_1, w_2 \in \mathbb{N}$ are pairwise coprime then the weighted projective plane $\mathbb{P}(w_0, w_1, w_2)$ has cyclic quotient singularities of types $\frac{1}{w_0}(w_1, w_2), \frac{1}{w_1}(w_2, w_0)$, and $\frac{1}{w_2}(w_0, w_1)$. Thus each weight w_i different from 1 corresponds to a singular point of X, and may be recovered as the order of the fundamental group of the link of the singular point.) If $r = 2$ then there are exactly two possibilities for a_3, related by the mutation $a_3' = 3a_1a_2 - a_3$. By Example 6.3 below these two choices yield isomorphic surfaces.

Finally suppose $r = 1$. Let $\frac{1}{n^2}(1, na - 1)$ be the isomorphism type of the singularity $P \in X$. Thus $n = a_1$ and $\frac{1}{n^2}(1, na-1) \simeq \frac{1}{a_1^2}(a_2^2, a_3^2)$. Equivalently, using the Markov equation,

$$\pm a = ((a_2^2 + a_3^2)/a_1) \cdot (a_2^2)^{-1} = (3a_2a_3 - a_1) \cdot (a_2^2)^{-1} = 3a_2^{-1}a_3 \bmod n \tag{6.3}$$

(the sign ambiguity comes from interchanging the orbifold coordinates). By inductively replacing (a_1, a_2, a_3) by a mutation at a_2 or a_3 (and appealing to Example 6.3 again), we may assume that $a_1 = \max(a_1, a_2, a_3)$, cf. [C57, p. 27]. Now by [R89, 3.2] (a_1, a_2, a_3) is uniquely determined by n and $\pm a \bmod n$. □

Example 6.3. Here we describe a two parameter family of surfaces which "connects" the weighted projective planes $\mathbb{P} := \mathbb{P}(a^2, b^2, c^2)$, $\mathbb{P}' := \mathbb{P}(a^2, b^2, c'^2)$ associated to two solutions of the Markov equation related by a single mutation. The family is given by

$$\mathcal{X} = (XY = sZ^{c'} + tT^c) \subset \mathbb{P}(a^2, b^2, c, c') \times \mathbb{A}^2_{s,t}.$$

(Note that $cc' = a^2 + b^2$ by (6.2).) The special fiber $X := \mathcal{X}_0$ is the union of two weighted projective planes $\mathbb{P}(a^2, c, c')$, $\mathbb{P}(b^2, c, c')$ glued along the coordinate lines of degree a^2 and b^2. It has two Wahl singularities of indices a and b and two orbifold normal crossing singularities of indices c and c'. The fibers $\mathcal{X}_{s,t}$ for $s = 0$, $t \neq 0$ are isomorphic to $\mathbb{P} = \mathbb{P}(a^2, b^2, c^2)$, with the embedding being the c-uple embedding

$$\mathbb{P}(a^2, b^2, c^2) \to (XY = T^c) \subset \mathbb{P}(a^2, b^2, c, c')$$

$$(U, V, W) \mapsto (X, Y, Z, T) = (U^c, V^c, W, UV).$$

Similarly, the fibers $\mathcal{X}_{s,t}$ for $s \neq 0$, $t = 0$ are isomorphic to $\mathbb{P}' = \mathbb{P}(a^2, b^2, c'^2)$. The fibers $\mathcal{X}_{s,t}$ for $s \neq 0$, $t \neq 0$ are obtained from $\mathbb{P}$ or $\mathbb{P}'$ by smoothing the singularity of index c or c' respectively. Moreover in each case the smoothing of the singularity is a versal $\mathbb{Q}$-Gorenstein deformation (cf. §2).

Theorem 6.4. *Let S denote the set of isomorphism classes of normal surfaces X such that X has a unique singular point $P \in X$ which is a quotient singularity and X admits a smoothing to $\mathbb{P}^2$. (Then $(P \in X)$ is a Wahl singularity and the smoothing is necessarily $\mathbb{Q}$-Gorenstein.)*

Let T denote the set of isomorphism classes of exceptional vector bundles F on $\mathbb{P}^2$ of rank greater than 1 modulo the operations $F \mapsto F^\vee$ and $F \mapsto F \otimes L$ for L a line bundle on $\mathbb{P}^2$.

Then Theorem 1.1 defines a bijection of sets

$$\Phi \colon S \to T, \quad [X] \mapsto [F].$$

Proof. Let X be a surface as in the statement. The singularity $(P \in X)$ is a Wahl singularity by Theorem 6.1. Let $(P \in X)$ be of type $\frac{1}{n^2}(1, na-1)$.

The smoothing of X to $\mathbb{P}^2$ is automatically $\mathbb{Q}$-Gorenstein by [M91], §1, Corollary 5. Let F denote an associated exceptional bundle F on $Y = \mathbb{P}^2$ given by Theorem 1.1. Then $\mathrm{rk}(F) = n$ and $c_1(F) \cdot K_Y = \pm a \bmod n$. Let h denote the hyperplane class on $\mathbb{P}^2$. Then

$$3(c_1(F) \cdot h) = \pm a \bmod n, \tag{6.4}$$

and the slope $\mu(F) := (c_1(F) \cdot h)/\,\mathrm{rk}(F) \in \mathbb{Q}$ is uniquely determined modulo translation by $\mathbb{Z}$ and multiplication by ± 1. An exceptional vector bundle on $\mathbb{P}^2$ is uniquely determined by its slope [DLP85, 4.3]. It follows that F is uniquely determined up to $F \mapsto F \otimes L$ and $F \mapsto F^\vee$. Thus the map Φ is well defined.

By Proposition 6.2 the surface X is uniquely determined by the isomorphism type of its singularity, which is given by n and $\pm a \mod n$. This data is determined by $[F] \in T$ as above, so Φ is injective. If F is an exceptional vector bundle on $\mathbb{P}^2$, then there exists a Markov triple (a_1, a_2, a_3) such that $\mathrm{rk}(F) = a_1$ and $(c_1(F) \cdot h) = \pm a_2^{-1} a_3 \bmod a_1$ [R89, 3.2]. Let X be the surface obtained from $\mathbb{P}(a_1^2, a_2^2, a_3^2)$ by smoothing the singularities of index a_2 and a_3. Then by (6.4) and (6.3) we have $[F] = \Phi([X])$. So Φ is surjective. $\square$

Remark 6.5. If Y is a del Pezzo surface, we can show the following weaker result: every exceptional bundle F on Y is obtained by the construction of Theorem 1.1. The proof uses the classification of exceptional bundles on del Pezzo surfaces [KO95].

7. Background

7.1. **Reflexive sheaves.** Let X be a normal variety. For F a coherent sheaf on X we write $F^\vee := \mathcal{H}om(F, \mathcal{O}_X)$ for the dual of E. We say F is *reflexive* if the natural map $F \to F^{\vee\vee}$ is an isomorphism. Equivalently, F is reflexive if F is torsion-free and for any inclusion $i \colon U \subset X$ of an open subset with complement of codimension at least 2 we have $i_*(F|_U) = F$ [H80, 1.6]. For a coherent sheaf F we call $F^{\vee\vee}$ the *reflexive hull* of F. A torsion-free sheaf F is reflexive iff it is satisfies Serre's condition S_2 [H80, 1.3]. In particular, a torsion-free sheaf F on a normal surface X is reflexive iff it is Cohen–Macaulay.

Let X be a normal projective variety of dimension d. The first Chern class defines a map

$$c_1\colon \operatorname{Pic} X \to H^2(X,\mathbb{Z}).$$

It is an isomorphism if $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$ (by the exponential sequence). Let $\operatorname{Cl} X$ denote the *class group* of reflexive rank 1 sheaves on X modulo isomorphism. Equivalently, $\operatorname{Cl} X$ is the group of Weil divisors on X modulo linear equivalence. We write $\mathcal{O}_X(D)$ for the sheaf associated to a Weil divisor D. Then $\operatorname{Pic} X \subset \operatorname{Cl} X$ and we have the map

$$c_1\colon \operatorname{Cl} X \to H_{2d-2}(X,\mathbb{Z}), \quad \mathcal{O}_X(D) \mapsto [D].$$

which is compatible with $c_1\colon \operatorname{Pic} X \to H^2(X,\mathbb{Z})$ via

$$H^2(X,\mathbb{Z}) \to H_{2d-2}(X,\mathbb{Z}), \quad \alpha \mapsto [X] \cap \alpha.$$

Let X be a normal projective surface with only quotient singularities. For each singular point $P_i \in X$ let B_i denote the intersection of X with a small closed ball about P_i in some embedding, and L_i the boundary of B_i. Let X^o denote the complement of the interiors of the B_i. Then each B_i is contractible (see e.g. [L84, 2.A]) and X^o is a compact oriented smooth manifold with boundary $\cup L_i$. We have natural identifications

$$H^2(X^o,\mathbb{Z}) = H_2(X^o,\cup L_i,\mathbb{Z}) = H_2(X,\cup B_i,\mathbb{Z}) = H_2(X,\mathbb{Z})$$

given by Poincaré duality for manifolds with boundary, excision, and contractibility of the B_i. Now suppose in addition that $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$. Then

$$\operatorname{Pic} X = H^2(X,\mathbb{Z}) \subset \operatorname{Cl} X = H^2(X^o,\mathbb{Z}) = H_2(X,\mathbb{Z}),$$

see [K05], Proposition 4.11.

Let X be a normal variety of dimension d. The *canonical sheaf* ω_X is the reflexive hull of the top exterior power $\wedge^d \Omega_X$ of the sheaf Ω_X of Kähler differentials. The *canonical divisor class* K_X is the associated Weil divisor class. If X is projective then the canonical sheaf ω_X coincides with the dualising sheaf of X [H77, III.7], [KM98, 5.75].

Let X be a normal projective surface and F a reflexive sheaf on X. Then, since F is Cohen–Macaulay, there is a natural isomorphism

$$H^i(X,F) \xrightarrow{\sim} H^{2-i}(X,\mathcal{H}om(F,\omega_X))^*$$

for each i given by Grothendieck–Serre duality [KM98, 5.71].

Lemma 7.1. *Let X be a projective normal surface with only quotient singularities and F a locally free sheaf on X. Then we have the Riemann–Roch formula*

$$\chi(F) = \operatorname{rk}(F)\chi(\mathcal{O}_X) + \frac{1}{2}c_1(F)(c_1(F) - K_X) - c_2(F).$$

Proof. Let $\pi\colon \tilde{X} \to X$ be the minimal resolution of X and $\tilde{F} = \pi^* F$. Then we have the Riemann–Roch formula for $\tilde{F}$ on $\tilde{X}$:

$$\chi(\tilde{F}) = \operatorname{rk}(\tilde{F})\chi(\mathcal{O}_{\tilde{X}}) + \frac{1}{2}c_1(\tilde{F})(c_1(\tilde{F}) - K_{\tilde{X}}) - c_2(\tilde{F}).$$

Note that $R^i\pi_*\mathcal{O}_{\tilde{X}} = 0$ for $i > 0$ (because quotient singularities are rational). Thus $\chi(\mathcal{O}_{\tilde{X}}) = \chi(\mathcal{O}_X)$ and $\chi(\tilde{F}) = \chi(F)$ by the Leray spectral sequence. Also $c_i(\tilde{F}) = \pi^* c_i(F)$ for each i, so $c_1(\tilde{F})^2 = c_1(F)^2$, $c_2(\tilde{F}) = c_2(F)$, and

$$c_1(\tilde{F}) \cdot K_{\tilde{X}} = \pi^* c_1(F) \cdot K_{\tilde{X}} = c_1(F) \cdot \pi_* K_{\tilde{X}} = c_1(F) \cdot K_X.$$

Combining we obtain the desired Riemann–Roch formula for F on X. □

7.2. **Toric geometry.** We use various constructions from toric geometry which we review briefly here. We refer to [F93] for the basic definitions of toric geometry.

7.2.1. *Weighted projective space.* The *weighted projective space* $\mathbb{P}(w_0, \ldots, w_r)$ is defined as the quotient

$$\mathbb{P}(w_0, \ldots, w_r) := (\mathbb{A}^{r+1} \setminus \{0\})/\mathbb{C}^\times$$

where the action is given by

$$\mathbb{C}^\times \ni \lambda\colon (x_0, \ldots, x_r) \mapsto (\lambda^{w_0} x_0, \ldots, \lambda^{w_r} x_r).$$

Then $\mathbb{P} = \mathbb{P}(w_0, \ldots, w_r)$ is a normal projective variety of dimension r. We may assume that any r of the w_i have no common factors. We have weighted homogeneous coordinates $X_0, \ldots, X_r$. The variety $\mathbb{P}$ is covered by affine patches

$$(X_i \neq 0) = \mathbb{A}^r/\tfrac{1}{w_i}(w_0, \ldots, \hat{w_i}, \ldots, w_r).$$

The variety $\mathbb{P}$ is the toric variety associated to the free abelian group $N = \mathbb{Z}^{r+1}/\mathbb{Z} \cdot (w_0, \ldots, w_r)$ and the fan Σ in $N \otimes \mathbb{R}$ of cones generated by proper subsets of the standard basis of $\mathbb{Z}^{r+1}$.

The variety $\mathbb{P}$ carries a rank 1 reflexive sheaf $\mathcal{O}_{\mathbb{P}}(1)$ such that the global sections of its nth reflexive tensor power $\mathcal{O}_{\mathbb{P}}(n) := (\mathcal{O}_{\mathbb{P}}(1)^{\otimes n})^{\vee\vee}$ are the weighted homogeneous polynomials of degree n. The class group $\operatorname{Cl}(\mathbb{P})$ of rank 1 reflexive sheaves modulo isomorphism is isomorphic to $\mathbb{Z}$, generated by $\mathcal{O}_{\mathbb{P}}(1)$. The canonical sheaf $\omega_{\mathbb{P}}$ is isomorphic to $\mathcal{O}_{\mathbb{P}}(-\sum w_i)$. The first Chern class $c_1\colon \operatorname{Cl}(\mathbb{P}) \to H_{2r-2}(\mathbb{P}, \mathbb{Z})$ is an isomorphism. Write $h := c_1(\mathcal{O}_{\mathbb{P}}(1))$ for the positive generator of $H_{2r-2}(\mathbb{P}, \mathbb{Z})$. Then the intersection product is given by $h^r = 1/(w_0 \cdots w_r)$.

7.2.2. *Weighted blowups.* Consider the cyclic quotient

$$X = \mathbb{A}^d_{x_1, \ldots, x_d}/\tfrac{1}{r}(a_1, \ldots, a_d).$$

The variety X is the affine toric variety associated to the free abelian group

$$N = \mathbb{Z}^d + \mathbb{Z} \cdot \tfrac{1}{r}(a_1, \ldots, a_d)$$

and the cone $\sigma \subset N \otimes \mathbb{R}$ generated by the standard basis of $\mathbb{Z}^d$. (That is, writing $M = \operatorname{Hom}(N, \mathbb{Z})$ and $\sigma^* \subset M \otimes \mathbb{R}$ for the dual cone, the semigroup ring $\mathbb{C}[\sigma^* \cap M]$ is the invariant ring for the action of $\mathbb{Z}/r\mathbb{Z}$ on $\mathbb{C}[x_1, \ldots, x_d]$.)

Let $w = \frac{1}{r}(w_1, \ldots, w_d) \in N$ be a primitive vector contained in the interior of the cone σ. Let $\tilde{\Sigma}$ be the fan with support σ obtained by adding the ray $\mathbb{R}_{\geq 0} \cdot w$ and subdividing σ into the cones spanned by w and the codimension 1 faces of σ. Then the fan $\tilde{\Sigma}$ determines a proper birational toric morphism

$$\pi \colon \tilde{X} \to X$$

called the *weighted blowup* of $P \in X$ with weights $\frac{1}{r}(w_1, \ldots, w_d)$ with respect to the orbifold coordinates $x_1, \ldots, x_d$. The morphism π restricts to an isomorphism over $X \setminus \{P\}$, and the exceptional locus $E = \pi^{-1}(P)$ is a quotient of the weighted projective space $\mathbb{P}(w_1, \ldots, w_d)$ by the action of a finite abelian group. The toric variety $\tilde{X}$ is covered by affine charts $U_1, \ldots, U_d$ corresponding to the maximal cones of Σ.

Assume for simplicity that w generates $N/\mathbb{Z}^d$. Then $E = \mathbb{P}(w_1, \ldots, w_d)$, and the restriction of π to the chart U_1 is given by

$$\mathbb{A}^d_{u, x_2', \ldots, x_d'} / \tfrac{1}{w_1}(-r, w_2, \ldots, w_d) \to \mathbb{A}^d_{x_1, \ldots, x_d} / \tfrac{1}{r}(a_1, \ldots, a_d),$$

$$(u, x_2', \ldots, x_d') \mapsto (x_1, \ldots, x_d) = (u^{w_1/r}, u^{w_2/r} x_2', \ldots, u^{w_d/r} x_d').$$

The other charts are described similarly.

7.3. **Deformation theory.**

Lemma 7.2. *Let X be a normal projective surface. Let $\mathcal{X}/(0 \in D)$ be the formal versal deformation of X and for each singularity $P_i \in X$ let $(P_i \in \mathcal{X}_i)/(0 \in D_i)$ be its formal versal deformation. Then there is a morphism (not unique)*

$$f \colon (0 \in D) \to \prod (0 \in D_i)$$

of formal schemes such that the deformation $(P_i \in \mathcal{X})/D$ of $(P_i \in X)$ is isomorphic to the pullback of the deformation $(P_i \in \mathcal{X}_i)/(0 \in D_i)$. If $H^2(T_X) = 0$ then f is formally smooth of relative dimension $\dim H^1(T_X)$. *In particular if $H^1(T_X) = H^2(T_X) = 0$ then f is an isomorphism.*

Proof. Let $L^{\cdot}_X$ denote the cotangent complex of X. Then the deformation functor of X has tangent space $\operatorname{Ext}^1(L^{\cdot}_X, \mathcal{O}_X)$ and obstruction space $\operatorname{Ext}^2(L^{\cdot}_X, \mathcal{O}_X)$, and the deformation functor of the singularities of X has tangent space $H^0(\mathcal{E}xt^1(L^{\cdot}_X, \mathcal{O}_X))$ and obstruction space $H^0(\mathcal{E}xt^2(L^{\cdot}_X, \mathcal{O}_X))$ [I71, III.2.1.7]. The cotangent complex $L^{\cdot}_X$ satisfies $L^i_X = 0$ for $i > 0$ and $\mathcal{H}^0(L^{\cdot}_X) = \Omega_X$, so $\mathcal{H}om(L^{\cdot}_X, \mathcal{O}_X) = \mathcal{H}om(\Omega_X, \mathcal{O}_X) = T_X$. Thus $H^i(\mathcal{H}om(L^{\cdot}_X, \mathcal{O}_X)) = H^i(T_X)$, and $H^2(T_X) = 0$ by assumption. The sheaf $\mathcal{E}xt^1(L^{\cdot}_X, \mathcal{O}_X)$ is supported on the singular locus of X, which is a finite set because X is a normal surface. Thus $H^i(\mathcal{E}xt^1(L^{\cdot}_X, \mathcal{O}_X)) = 0$ for $i > 0$. Now the local-to-global spectral sequence for $\operatorname{Ext}^{\cdot}(L^{\cdot}_X, \mathcal{O}_X)$ gives a short exact sequence

$$0 \to H^1(T_X) \to \operatorname{Ext}^1(L^{\cdot}_X, \mathcal{O}_X) \to H^0(\mathcal{E}xt^1(L^{\cdot}_X, \mathcal{O}_X)) \to 0$$

and an isomorphism

$$\operatorname{Ext}^2(L^{\cdot}_X, \mathcal{O}_X) \to H^0(\mathcal{E}xt^2(L^{\cdot}_X, \mathcal{O}_X)).$$

It follows that $f\colon (0 \in D) \to \prod(0 \in D_i)$ is formally smooth with relative tangent space $H^1(T_X)$. □

7.4. **Glueing vector bundles on reducible schemes.**

Lemma 7.3. *Let X be a connected reduced scheme with two irreducible components X_1 and X_2. Let X_{12} be the scheme theoretic intersection of X_1 and X_2 in X. Suppose given vector bundles F_1 on X_1, F_2 on X_2, and F_{12} on X_{12}, and isomorphisms*

$$\theta_1\colon F_1|_{X_{12}} \to F_{12}, \quad \theta_2\colon F_2|_{X_{12}} \to F_{12}.$$

Let F denote the kernel of the morphism

$$F_1 \oplus F_2 \to F_{12}, \quad (s_1, s_2) \mapsto \theta_1(s_1|_{X_{12}}) - \theta_2(s_2|_{X_{12}}).$$

of sheaves on X. Then F is a vector bundle on X such that the projection $F|_{X_i} \to F_i$ is an isomorphism for each i.

We refer to F as the vector bundle on X obtained by glueing F_1 on X_1 and F_2 on X_2 along F_{12} on X_{12}.

Proof. We have an exact sequence of sheaves on X

$$0 \to \mathcal{O}_X \to \mathcal{O}_{X_1} \oplus \mathcal{O}_{X_2} \to \mathcal{O}_{X_{12}} \to 0.$$

We may work locally at a point $P \in X_{12} \subset X$. Let $s_1, \ldots, s_r$ be a basis of the free $\mathcal{O}_{X_{12},P}$-module $(F_{12})_P$. Lift this basis to a basis $s^i_1, \ldots, s^i_r$ of the free $\mathcal{O}_{X_i,P}$-module $(F_i)_P$ for each $i = 1, 2$. Then we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_{X,P}^{\oplus r} & \longrightarrow & \mathcal{O}_{X_1,P}^{\oplus r} \oplus \mathcal{O}_{X_2,P}^{\oplus r} & \longrightarrow & \mathcal{O}_{X_{12},P}^{\oplus r} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_P & \longrightarrow & (F_1)_P \oplus (F_2)_P & \longrightarrow & (F_{12})_P & \longrightarrow & 0 \end{array}$$

such that the middle and right vertical arrows are the isomorphisms given by the chosen bases. Hence the left vertical arrow is an isomorphism. So F is locally free, and the morphism $F|_{X_i} \to F_i$ is an isomorphism for each i. □

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