STABLE PAIR, TROPICAL, AND LOG CANONICAL COMPACTIFICATIONS OF MODULI SPACES OF DEL PEZZO SURFACES

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ABSTRACT. We give a functorial normal crossing compactification of the moduli space of smooth cubic surfaces entirely analogous to the Grothendieck-Knudsen compactification $M_{0,n} \subset \overline{M}_{0,n}$.

§1. INTRODUCTION AND STATEMENT OF RESULTS

Throughout we work over an algebraically closed field k. Let Y^n be the moduli space of smooth marked del Pezzo surfaces S of degree $9-n$. We begin by observing that each such surface comes with a natural boundary: The marking of S induces a labelling of its (-1)-curves, and we can take their union $B \subset S$. This gives a natural way of compactifying the space: Let $Y^n_{\times} \subset Y^n$ denote the open locus where B has normal crossings. For $\operatorname{char} k = 0$, let $\overline{\mathcal{M}}$ denote the Kollár–Shepherd-Barron–Alexeev moduli stack of stable surfaces with boundary [KS, Al3, Ko6] and $\overline{M}$ its coarse moduli space. (See 10.16 for the precise definition of $\overline{\mathcal{M}}$.) We can compactify $Y^n_{\times}$ by taking its closure in $\overline{M}$. This closure turns out to have very nice properties:

1.1. THEOREM. *Assume $n \leq 5$ or $n = 6$ and $\operatorname{char} k \neq 2$. There is a compactification $Y^n \subset \overline{Y}^n_{\mathrm{ss}}$ and a family $p\colon (\mathcal{S}, \mathcal{B}) \to \overline{Y}^n_{\mathrm{ss}}$ of stable surfaces with boundary extending the universal family of smooth del Pezzo surfaces with normal crossing boundary over $Y^n_{\times}$. $\overline{Y}^n_{\mathrm{ss}}$ is a smooth projective variety, and $\overline{Y}^n_{\mathrm{ss}} \setminus Y^n$ is a union of smooth divisors with normal crossings. The fibers of p have stable toric singularities. If $\operatorname{char} k = 0$, p defines a closed embedding $\overline{Y}^n_{\mathrm{ss}} \subset \overline{M}$.*

(Recall that a pair (S, B) has *stable toric singularities* if it is locally isomorphic to a stable toric variety [Al4, §1.1A] together with its toric boundary. A stable toric variety is obtained by glueing normal toric varieties along toric boundary divisors, and its toric boundary is the union of the remaining boundary divisors.)

We call the fibres of $p\colon (\mathcal{S}, \mathcal{B}) \to \overline{Y}^n_{\mathrm{ss}}$ stable del Pezzo pairs. We will show that each fibre (S, B) comes with a canonical embedding in a stable toric pair (X, B), with $S \subset X$ transverse to the toric strata (with the boundary of S the restriction of the toric boundary), see Theorems 1.18 and 1.19.

For a smooth variety Y and $n \geq 0$ the vector space $H^0(\overline{Y}, n(K_{\overline{Y}} + B))$, for $Y \subset \overline{Y}$ a compactification with normal crossing boundary $B := \overline{Y} \setminus Y$, depends only on Y. Conjecturally the associated graded ring

$$R(\overline{Y}, K_{\overline{Y}} + B) := \bigoplus_{n \geq 0} H^0(\overline{Y}, n(K_{\overline{Y}} + B))$$

(called the *log canonical ring*) is finitely generated [KoM, 3.12]. When this holds the associated rational map $Y \dashrightarrow \overline{Y}_{\mathrm{lc}} := \operatorname{Proj} R(\overline{Y}, K_{\overline{Y}} + B)$ is called the *log canonical model*. Y is called *log minimal* if for some n the natural rational map $Y \dashrightarrow \mathbb{P}(H^0(\overline{Y}, n(K_{\overline{Y}} + B))^*)$ is an embedding. In this case, if $R(\overline{Y}, K_{\overline{Y}} + B)$ is finitely generated, the log canonical model $Y \dashrightarrow \overline{Y}_{\mathrm{lc}}$ is regular and an embedding, and thus $Y \subset \overline{Y}_{\mathrm{lc}}$ is a natural compactification.

1.2. Theorem. *The variety Y^n is log minimal. For $n \leq 6$ or $n = 7$ and $\operatorname{char} k \neq 2$, the log canonical ring of Y^n is finitely generated, the variety $\overline{Y}^n_{\mathrm{lc}}$ is smooth, and $\overline{Y}^n_{\mathrm{lc}} \setminus Y^n$ is a union of smooth divisors with normal crossings.*

If S is a del Pezzo surface of degree $9-n$ and $C \subset S$ is a (-1)-curve, then the surface S' obtained by contracting C is a del Pezzo surface of degree $9-(n-1)$. Fixing a consistent choice of C using the marking, we obtain a natural morphism $\pi : Y^{n+1} \to Y^n$. Assume $n \leq 5$ or $n = 6$ and $\operatorname{char} k \neq 2$. Then π extends to a morphism $\pi : \overline{Y}^{n+1}_{\mathrm{lc}} \to \overline{Y}^n_{\mathrm{lc}}$ and we have a commutative diagram

$$\begin{array}{ccc} \mathcal{S} & \longrightarrow & \overline{Y}^{n+1}_{\mathrm{lc}} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ \overline{Y}^n_{\mathrm{ss}} & \longrightarrow & \overline{Y}^n_{\mathrm{lc}}, \end{array}$$

where the horizontal arrows are isomorphisms for $n \leq 5$ and log crepant birational morphisms for $n = 6$. Moreover, $\overline{Y}^6_{\mathrm{ss}} \to \overline{Y}^6_{\mathrm{lc}}$ is a composition of blowups with smooth centers. In particular $\overline{Y}^n_{\mathrm{ss}}$ is a log minimal model for Y^n and its universal family is a log minimal model for Y^{n+1}.

Note the close analogy with $M_{0,n} \subset \overline{M}_{0,n}$: This is the log canonical model as well as the closure in the moduli space of pointed stable curves, and $\overline{M}_{0,n+1} \to \overline{M}_{0,n}$ is the universal family. The two cases are instances of a single construction which we explain below.

1.3. Remarks.

(1) In general boundary strata of irreducible components of $\overline{\mathcal{M}}$ have arbitrary singularities, even for (degenerations of) lines in $\mathbb{P}^2$. Already for 9 lines the compactification has non log canonical boundary. See [KT, Thm. 1.5]. The del Pezzo case is thus highly exceptional.

(2) The Weyl group $W(E_n)$ acts on Y^n by changing the marking. Our constructions are equivariant, so we can quotient by $W(E_n)$ to get moduli spaces of unmarked del Pezzo surfaces, and orbifold versions of the above theorems.

(3) We expect that $\overline{Y}^n_{\mathrm{ss}}$ is a connected component of $\overline{M}$ and is a fine moduli space. We have verified this for $n = 5$ by explicit deformation theory calculations.

(4) Following the philosophy of [H] and [Al4] we believe that $\overline{Y}^n_{\mathrm{lc}}$ ($n \geq 6$) itself has a natural functorial meaning – namely the moduli space of stable pairs (S, cB) where the coefficient c is minimal. That is, let $\alpha \in \mathbb{Q}$ be such that $K_S + \alpha B = 0$ for a general fibre (S, B), and define $c = \alpha + \epsilon$ where $0 < \epsilon \ll 1$.

(5) The universal family $(\mathcal{S}, \mathcal{B}) \to \overline{Y}^n_{\mathrm{ss}}$ is highly non-trivial. For example, in the first non-constant case $(\mathcal{S}, \mathcal{B}) \to \overline{Y}^5_{\mathrm{ss}}$, the base is a copy of $\overline{M}_{0,5}$. The divisor $\mathcal{B}$ has 16 irreducible components, each a copy of the universal family $\overline{M}_{0,6} \to \overline{M}_{0,5}$. The simplest degenerate fibre is a surface with 6 (smooth) components (across which the 16 (-1)-curves are distributed).

(6) The tautological map $\pi : Y_{n+1} \to Y_n$ (given by blowing down a (-1)-curve) induces a map of tropical varieties $\operatorname{trop}(\pi) : \operatorname{trop} Y_{n+1} \to \operatorname{trop} Y_n$ — a map of fans induced by a linear map of ambient real vector spaces (we review tropical geometry below and give a new, and self contained, treatment of the theory in §2). $\operatorname{trop}(\pi)$ is canonically associated with the inclusion of

root systems $E_n \subset E_{n+1}$ as we explain below. Our compactifications, and the universal families, together with all the combinatorics of the boundary and fibres, can be read off from π and this map of fans: together they determine the functorial compactification, and the functor that it represents. In particular we get the complete picture of possible degenerations, without considering degenerations (or even surfaces) at all.

We will identify $\overline{Y}^n_{\mathrm{lc}}$ with known spaces: $\overline{Y}^5_{\mathrm{lc}} \simeq \overline{M}_{0,5}$, $\overline{Y}^6_{\mathrm{lc}}$ is isomorphic to the Naruki space $\overline{Y}^6$ of cubic surfaces [N], and $\overline{Y}^7_{\mathrm{lc}}$ is isomorphic to the Sekiguchi space $\overline{Y}^7$ of del Pezzo surfaces of degree 2 [S1, S2]. Sekiguchi constructed $\overline{M}_{0,n}$, $\overline{Y}^6$, and $\overline{Y}^7$ in a unified way, using "cross-ratio maps" associated to root subsystems of type D_4 in root systems D_n, E_6, and E_7.

1.4. Definition. Let S be a marked del Pezzo surface. By a *KSBA cross-ratio* we mean the cross-ratio of 4 points on a (-1)-curve L cut out by (-1)-curves $L_1, \dots, L_4$ (assuming that these points are distinct and that $L_i \cdot L = 1$). If, in addition, $L_i \cdot L_j = 0$ for $1 \le i < j \le 4$ then we have a *cross-ratio of type I*.

We show that KSBA cross-ratios of type I are the same as Sekiguchi's D_4 cross-ratios. Sekiguchi defines $\overline{Y}^n$ as the closure of the image of Y^n in the product $(\mathbb{P}^1)^N$ given by all cross-ratios of type I. Naruki proved that $\overline{Y}^6$ is smooth and its boundary has simple normal crossings. Sekiguchi conjectured the analogous result for $n = 7$. We prove:

1.5. Theorem (Sekiguchi Conjecture). *For $n \le 6$ or $n = 7$ and $\operatorname{char} k \ne 2$, $\overline{Y}^n = \overline{Y}^n_{\mathrm{lc}}$. In particular, $\overline{Y}^7$ is smooth and the boundary $\overline{Y}^7 \setminus Y^7$ has simple normal crossings.*

It is rather unnatural to take only cross-ratios of type I:

1.6. Theorem. *For $n \le 5$ or $n = 6$ and $\operatorname{char} k \ne 2$, the closure of the image of $Y^n_\times$ in the product $(\mathbb{P}^1)^M$ given by all KSBA cross-ratios is isomorphic to $\overline{Y}^n_{\mathrm{ss}}$.*

We obtain $\overline{Y}^n_{\mathrm{lc}}$, $\overline{Y}^n_{\mathrm{ss}}$, and its universal family $(\mathcal{S}, \mathcal{B})$, together with an ambient family of stable toric pairs $(\mathcal{X}, \mathcal{B})$, canonically from the interior Y^n by applying elementary ideas from tropical algebraic geometry [T]. The same construction applied to $M_{0,n}$ yields $\overline{M}_{0,n}$. Next we explain the procedure.

Let Y be a variety defined over an algebraically closed field k. The group of units $M := \mathcal{O}^*(Y)/k^*$ is a free Abelian group of finite rank. Let $N = \operatorname{Hom}(M, \mathbb{Z})$. The tropical variety $\mathcal{A}$ of Y is an intrinsic subset of $N_{\mathbb{Q}}$. We refer to [T, EKL, SS] for its definition. In §2 we will prove the following new characterization of $\mathcal{A}$:

$$\mathcal{A}(Y) = \big\{[v] \;\big|\; v \text{ is a discrete valuation of } k(Y)\big\} \subset N_{\mathbb{Q}}, \qquad (1.6.1)$$

where $[v](u) = v(u)$ for any unit $u \in M$. Note $[v] = 0$ if v has center on Y, thus by (1.6.1) $\mathcal{A}(Y)$ "sees" the boundary divisors from all compactifications of Y. We show it can be computed from a single normal crossing compactification, see Cor. 2.4.

1.7. Definition ([T, 1.1]). Let $T := \operatorname{Hom}(M, \mathbb{G}_m)$ be the *intrinsic* algebraic torus. Choosing a splitting of the exact sequence

$$0 \to k^* \to \mathcal{O}^*(Y) \to M \to 0$$

defines an evaluation map $Y \to T$ (any two such are related by a translation by an element of T). We say Y is *very affine* if $Y \to T$ is a closed embedding

Now suppose Y is very affine. Let $\mathcal{F} \subset N_{\mathbb{Q}}$ be a fan, $X(\mathcal{F})$ the corresponding toric variety, and $\overline{Y}(\mathcal{F})$ the closure of Y in $X(\mathcal{F})$. We say $\mathcal{F}$ is a *tropical fan* (for Y) and $\overline{Y}(\mathcal{F})$ is a *tropical compactification* if $\overline{Y}(\mathcal{F})$ is complete and the multiplication map $\overline{Y}(\mathcal{F}) \times T \to X(\mathcal{F})$ is flat and surjective.

1.8. Theorem ([T, 1.7, 2.3, 2.5]). *Let Y be a very affine variety. Then there exists a tropical fan for Y. If $\mathcal{F}$ is a tropical fan for Y then the support $|\mathcal{F}|$ of $\mathcal{F}$ is equal to the tropical variety $\mathcal{A}$ of Y. Any refinement of a tropical fan for Y is also a tropical fan for Y.*

It is natural to wonder if there is a canonical fan structure on $\mathcal{A}$. The answer is no in general, but there is a natural Mori-theoretic sufficient condition:

1.9. Definition. We say that a very affine variety Y is *Schön* if the multiplication map of one (and hence of any [T, 1.4]) tropical compactification is smooth. A Schön variety Y is called *Hübsch* if it is log minimal, has a log canonical model $\overline{Y}_{\mathrm{lc}}$, and $\overline{Y}_{\mathrm{lc}}$ is a tropical compactification of Y, i.e., $\overline{Y}_{\mathrm{lc}} \simeq \overline{Y}(\mathcal{F})$ for some fan $\mathcal{F}$ (called the *log canonical fan*).

In fact we prove that a Schön subvariety of an algebraic torus is either log minimal or preserved by a subtorus, see §3. This is an analogue of a well-known result for subvarieties of abelian varieties. The possibility of such an analogue was raised by Miles Reid.

1.10. Theorem. *Let Y be a Hübsch very affine variety. Then every fan structure on $\mathcal{A}$ is tropical, and a refinement of the log canonical fan.*

Thus in the Hübsch case $\mathcal{A} \subset N_{\mathbb{Q}}$ has a canonical fan structure, which yields the log canonical model embedded in a canonical ambient toric variety, and transverse to the toric boundary. The construction works in the other direction as well — $\mathcal{A}$ together with its canonical fan structure can be read off from the boundary stratification of the log canonical model, see §2.

Here are some examples of Hübsch varieties:

1.11. Theorem. *$M_{0,n}$ and Y^n (for $n \leq 6$ or $n = 7$ and $\operatorname{char} k \neq 2$) are Hübsch. The corresponding log canonical models are $\overline{M}_{0,n}$ and $\overline{Y}^n_{\mathrm{lc}}$.*

$M_{0,n}$ is isomorphic to the complement of the braid arrangement of hyperplanes in $\mathbb{P}^{n-3}$. More generally, recall that a hyperplane arrangement in $\mathbb{P}^m$ is *connected* if the subgroup of PGL_{m+1} preserving the arrangment is finite (or equivalently, if the complement is log minimal). By [HKT, §2]:

1.12. Theorem. *Complements of arbitrary connected hyperplane arrangments are Hübsch and their log canonical models are Kapranov's visible contours* [Kap].

We write $Y(E_n) := Y^n$ and $Y(D_n) := M_{0,n}$. We will treat the two cases in a unified way. Δ will indicate a root system of type D_n or E_n. Next we describe the log canonical fans for $Y(\Delta)$ using combinatorics of root subsystems introduced by Sekiguchi [S2]. Let $W(\Delta)$ be the Weyl group. Let $\Delta_+ \subset \Delta$ be the positive roots. $W(D_n)$ acts on $M_{0,n}$ through its quotient S_n and $W(E_n)$ acts on Y^n by changing markings, see [DO, Ch. V] or §6 for details.

1.13. Theorem. *Let $\mathbb{Z}^{\Delta_+}$ be the lattice with basis vectors $[\alpha]$ for $\alpha \in \Delta_+$. Let*

$$M(\Delta) = \left\{ \sum n_\alpha [\alpha] \;\middle|\; \sum n_\alpha \alpha^2 = 0 \right\}. \tag{1.13.1}$$

Then $M(\Delta)$ is an irreducible $W(\Delta)$-module of rank equal to the number of roots in Δ_+ with three-legged support. We have an isomorphism of $W(\Delta)$-modules

$$\mathcal{O}^*(Y(\Delta))/k^* = M(\Delta).$$

1.14. Let $\psi : \mathbb{Z}^{\Delta_+} \to N(\Delta)$ be the dual map. For any root subsystem $J \subset \Delta$, let

$$\psi(J) := \sum_{\alpha \in J \cap \Delta_+} \psi(\alpha)$$

and let $\mathcal{J}$ be the set of root subsystems of Δ of type J. For example, $\mathcal{A}_1$ is the set of A_1's in Δ. Let $\mathcal{R}(\Delta)$ be a simplicial complex defined as follows. As a set,

$$\mathcal{R}(D_{2n+1}) = \mathcal{D}_2 \sqcup \mathcal{D}_3 \sqcup \ldots \sqcup \mathcal{D}_n;$$

$$\mathcal{R}(D_{2n}) = \mathcal{D}_2 \sqcup \mathcal{D}_3 \sqcup \ldots \sqcup \mathcal{D}_{n-1} \sqcup (\mathcal{D}_n \times \mathcal{D}_n);$$

$$\mathcal{R}(E_6) = \mathcal{A}_1 \sqcup (\mathcal{A}_2 \times \mathcal{A}_2 \times \mathcal{A}_2);$$

$$\mathcal{R}(E_7) = \mathcal{A}_1 \sqcup \mathcal{A}_2 \sqcup (\mathcal{A}_3 \times \mathcal{A}_3) \sqcup \mathcal{A}_7.$$

Subsystems $\Theta_1, \ldots, \Theta_k \in \mathcal{R}$ form a simplex if and only if $\Theta_i \perp \Theta_j$, or $\Theta_i \subset \Theta_j$, or $\Theta_j \subset \Theta_i$ for any pair i, j with the following exception: For $\Delta = E_7$, we exclude "Fano simplices" formed by 7-tuples of pairwise orthogonal A_1's.

Let $\mathcal{F}(\Delta) \subset N(\Delta)_{\mathbb{Q}}$ be the collection of cones determined by the rays $\psi(\Theta)$ for each $\Theta \in \mathcal{R}(\Delta)$, where rays span a cone in $\mathcal{F}(\Delta)$ iff the corresponding subsystems form a simplex in $\mathcal{R}(\Delta)$.

1.15. Definition. We say that a collection $\{\sigma_i\}_{i \in I}$ of convex subsets of $\mathbb{R}^n$ is *convexly disjoint* if any convex subset of $\bigcup_{i \in I} \sigma_i$ is contained in σ_j for some $j \in I$.

For N a free abelian group of finite rank and σ a cone in $N \otimes \mathbb{Q}$, we say σ is *strictly simplicial* if it is generated by a subset of a basis of N. We say a fan $\mathcal{F}$ is strictly simplicial if each cone of $\mathcal{F}$ is so. Equivalently, the associated toric variety $X(\mathcal{F})$ is smooth.

1.16. Theorem. *($n \leq 6$ or $n = 7$ and $\operatorname{char} k \neq 2$ for E_n) $\mathcal{F}(\Delta)$ is the log canonical fan of $Y(\Delta)$. This fan is convexly disjoint and strictly simplicial.*

In what follows we frequently use

1.17. Theorem ([T, 3.1]). *A dominant map of very affine varieties $Y' \to Y$ induces a surjective homomorphism of intrinsic tori $T_{Y'} \to T_Y$ and a surjective map of tropical varieties $\mathcal{A}(Y') \to \mathcal{A}(Y)$.*

Note that if Y is Hübsch and the log canonical fan is strictly convex, then $\mathcal{A}(Y') \to \mathcal{A}(Y)$ is a map of fans, for any fan structure on $\mathcal{A}(Y')$ (and the log canonical fan structure on $\mathcal{A}(Y)$).

1.18. Theorem. *Let (Δ_{n+1}, Δ_n) be either (D_{n+1}, D_n) or (E_{n+1}, E_n) for $n \leq 5$ or $n = 6$ and $\operatorname{char} k \neq 2$. The natural morphism $Y(\Delta_{n+1}) \to Y(\Delta_n)$ induces a surjective map of fans $\mathcal{F}(\Delta_{n+1}) \to \mathcal{F}(\Delta_n)$. This induces a commutative diagram:*

$$\begin{array}{ccc} \overline{Y}_{\mathrm{lc}}(\Delta_{n+1}) & \longrightarrow & X(\mathcal{F}(\Delta_{n+1})) \\ {\scriptstyle p}\downarrow & & {\scriptstyle \pi}\downarrow \\ \overline{Y}_{\mathrm{lc}}(\Delta_n) & \longrightarrow & X(\mathcal{F}(\Delta_n)) \end{array}$$

where the horizontal maps are closed embeddings. For $\Delta_n = D_n$, or E_n, $n \leq 5$, π is a flat family of stable toric varieties, and p is a flat family of stable pairs which induces an isomorphism $\overline{Y}_{\mathrm{lc}}(\Delta_n) \xrightarrow{\sim} \overline{M}_{0,n}$ for D_n and a closed embedding of $\overline{Y}_{\mathrm{lc}}(\Delta_n)$ into the moduli space $\overline{M}$ of stable surfaces with boundary for E_n (assuming $\operatorname{char} k = 0$). Each fibre of p is transverse to the toric strata of the corresponding fibre of π.

If $\Delta_n = E_6$ then π is not flat. However there is a canonical combinatorial procedure for flattening a toric morphism. We write $\mathcal{F}^n$ for $\mathcal{F}(E_n)$.

1.19. Theorem. *Assume $\operatorname{char} k \neq 2$. There is a unique minimal refinement $\tilde{\mathcal{F}}^6$ of $\mathcal{F}^6$ such that the closure $X(\tilde{\mathcal{F}}^7)$ of the torus $T(Y^7)$ in the fibre product $X(\mathcal{F}^7) \times_{X(\mathcal{F}^6)}$*

$X(\tilde{\mathcal{F}}^6)$ is flat over $X(\tilde{\mathcal{F}}^6)$. $X(\tilde{\mathcal{F}}^7)$ is a normal toric variety, let $\tilde{\mathcal{F}}^7 \subset N(E_7)$ be the corresponding fan. There is an induced commutative diagram

$$\begin{array}{ccccc} \overline{Y}(\tilde{\mathcal{F}}^7) & \xrightarrow{i} & X(\tilde{\mathcal{F}}^7) & \longrightarrow & X(\mathcal{F}^7) \\ {\scriptstyle \tilde{p}}\downarrow & & {\scriptstyle \tilde{\pi}}\downarrow & & {\scriptstyle \pi}\downarrow \\ \overline{Y}(\tilde{\mathcal{F}}^6) & \xrightarrow{j} & X(\tilde{\mathcal{F}}^6) & \longrightarrow & X(\mathcal{F}^6) \end{array}$$

where i and j are closed embeddings. $\tilde{p}$ and $\tilde{\pi}$ are flat with reduced fibers. $X(\tilde{\mathcal{F}}^6)$ and $\overline{Y}(\tilde{\mathcal{F}}^6)$ are smooth with simple normal crossing boundary. Away from Eckhart points, each fibre of $\tilde{p}$ is transverse to the toric strata of the corresponding fibre of $\tilde{\pi}$. Blowing up the Eckhart points of $\tilde{p}$ yields a family of stable surfaces with boundary which induces a closed embedding of $\overline{Y}(\tilde{\mathcal{F}}^6)$ in $\overline{M}$ for $\operatorname{char} k = 0$.

Recall that an *Eckhart point* on a smooth cubic surface S is an ordinary triple point of the union B of the (-1)-curves. The Eckhart points of the family $\tilde{p}$ are by definition the closure of the locus of Eckhart points on the smooth fibres. If s is an Eckhart point and (S, B) is the fiber through s, then S is smooth near s and B has an ordinary triple point at s. See 10.21.

Y^n has received considerable attention from the point of view of hyperplane arrangements on ball quotients, by Kondō, Dolgachev, Heckman, Looijenga and others. See [L] and the references there. In particular Y^n has a Baily-Borel compactification, and Looijenga gives desingularisations which are naturally determined by the space together with its arrangement of hyperplanes. $\overline{Y}^n_{\mathrm{lc}}$, $n \leq 7$ can be constructed by Looijenga's procedure. None of these authors (to our knowledge) consider the modular meaning of the compactification (i.e. whether the boundary parameterizes any sort of geometric object).

1.20. Outline of the proof. Let $Y = Y(\Delta)$. By Cor. 2.4, to find $\mathcal{A}(Y)$, expressed as the union of a collection of cones $\mathcal{F}$ (which may or may not form a fan), it is enough to find an orbifold normal crossing compactification and describe its boundary. We use the compactification $\overline{Y}_{\mathrm{lc}}$ (which we show has orbifold normal crossings). We identify $\overline{Y}_{\mathrm{lc}}$ for D_n with $\overline{M}_{0,n}$ and $\overline{Y}_{\mathrm{lc}}$ for E_6 with the Naruki space $\overline{Y}^6$, see Cor. 9.2. For E_7 there is a straightforward way to compute $\overline{Y}_{\mathrm{lc}}$: The anticanonical map for a smooth del Pezzo of degree 2 is a double cover of $\mathbb{P}^2$ branched over a smooth quartic. This expresses Y as a finite Galois cover $Y \to M_3 \setminus H$ of the moduli space of smooth non-hyperelliptic curves of genus 3. The log canonical model of Y is then the normalization in this field extension of the log canonical model of $M_3 \setminus H$. The latter turns out to be not $\overline{M}_3$ but the blowup $\tilde{M}_3 \to \overline{M}_3$ along the locus of curves whose generic point corresponds to two genus 1 curves glued at two points.

Using root systems we can describe $\mathcal{F}$ and prove that it is a strictly simplicial fan and convexly disjoint by purely combinatorial means. This in particular implies that $\overline{Y}_{\mathrm{lc}}$ is smooth, the boundary has simple normal crossings, and that $\overline{Y}_{\mathrm{lc}}$ embeds in the toric variety with the fan $\mathcal{F}$. This is carried out in sections 4–8. Our main technical tool is a (it seems to us rather miraculous) larger more symmetric fan $\mathcal{G}$ coming from the real points of Y, see §8. The dominant tautological map $Y^{n+1} \to Y^n$ induces a surjection of sets $\mathcal{A}(Y^{n+1}) \to \mathcal{A}(Y^n)$, which is a map of fans $\mathcal{F}^{n+1} \to \mathcal{F}^n$, as $\mathcal{F}^n$ is convexly disjoint. Therefore, there is an induced morphism of toric varieties, and thus of the log canonical models $\overline{Y}^{n+1}_{\mathrm{lc}} \to \overline{Y}^n_{\mathrm{lc}}$. For $n \leq 5$ this is the universal family of stable surfaces. For $n = 6$ the map of toric varieties is not flat, but the canonical procedure for flattening such a map leads to the universal family. See §10.

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CONTENTS

§2. GEOMETRIC TROPICALIZATION AND COX COORDINATES

Let K be the field of Puiseux series over k and R its valuation ring. The valuation deg induces an isomorphism $K^*/R^* \simeq \mathbb{Q}$. Let Y be an affine variety over K, $M = \mathcal{O}^*(Y)/K^*$, and $\tilde{M} := \mathcal{O}^*(Y)/R^*$. We have an exact sequence

$$0 \to K^*/R^* \to \tilde{M} \to M \to 0.$$

2.1. DEFINITION. For a field extension $K \subset L$, a map $f : \operatorname{Spec}(L) \to Y$, and a valuation $v : L^* \to \mathbb{Q}$ such that $v(R^*) = 0$ and $v(R) \geq 0$, let $[f, v] \in \tilde{N}_{\mathbb{Q}} := \operatorname{Hom}(\tilde{M}, \mathbb{Q})$ be the element induced by the composition $\mathcal{O}^*(Y) \xrightarrow{f^*} L^* \xrightarrow{v} \mathbb{Q}$. Let $\tilde{\mathcal{A}} \subset \tilde{N}_{\mathbb{Q}}$ be the union of such $[f, v]$. The *Bieri-Groves set* [BG, EKL] $\mathcal{A} \subset \tilde{\mathcal{A}}$ is the union of $[f, v]$ such that the restriction of v to R is the standard valuation.

2.2. DEFINITION. Let $\overline{Y} \to \operatorname{Spec} R$ be a normal $\mathbb{Q}$-factorial variety of finite type over $\operatorname{Spec} R$. Let $Y \subset \overline{Y}$ be an open subset such that the boundary $\partial\overline{Y} := \overline{Y} \setminus Y$ is divisorial. Let $\mathcal{F}(\partial\overline{Y})$ denote the collection of cones in $\tilde{N}(Y)_{\mathbb{Q}}$ defined as follows: For each irreducible boundary divisor $D \subset \partial\overline{Y}$ let ord_D be the corresponding valuation of the function field $K(Y)$. This is trivial on R^* (since elements of R^* pull back to units on $\overline{Y}$) and so induces an element $[D] \in \tilde{N}_{\mathbb{Q}}$. For any collection $\sigma = \{D_1, \ldots, D_k\} \subset \partial\overline{Y}$ such that $\bigcap D_i \neq \emptyset$, let $\mathcal{F}(\sigma)$ be the cone in $\tilde{N}_{\mathbb{Q}}$ spanned by $[D_1], \ldots, [D_k]$.

If $\overline{Z}$ is a normal $\mathbb{Q}$-factorial variety over k and $Z \subset \overline{Z}$ an open subset with divisorial boundary $\partial\overline{Z} := \overline{Z} \setminus Z$, we define a collection of cones $\mathcal{F}(\partial\overline{Z}) \subset N(Z)_{\mathbb{Q}}$, where $N(Z) = \operatorname{Hom}(M(Z), \mathbb{Z})$ and $M(Z) = \mathcal{O}^*(Z)/k^*$, in the same way as above. That is, the rays of $\mathcal{F}(\partial\overline{Z})$ are generated by the valuations ord_D for D a component of $\partial\overline{Z}$, and a collection of rays spans a cone of $\mathcal{F}(\partial\overline{Z})$ if the corresponding divisors intersect.

A version of the following theorem was proved by W. Gubler [G, 7.10]. We observed the result independently in joint work with Z. Qu [Q].

2.3. THEOREM. *Suppose we have a diagram*

$$\begin{array}{ccc} \mathcal{Y} & \longrightarrow & \overline{\mathcal{Y}} \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} R \end{array}$$

where $\mathcal{Y} \subset \overline{\mathcal{Y}}$ is an open substack of a smooth Deligne-Mumford stack, proper over $\operatorname{Spec}(R)$*, such that the boundary $\partial\overline{\mathcal{Y}} := \overline{\mathcal{Y}} \setminus \mathcal{Y}$ is a divisor with simple normal crossings. Let*

$$\begin{array}{ccc} Y & \longrightarrow & \overline{Y} \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & \operatorname{Spec} R \end{array}$$

be the induced diagram of coarse moduli spaces — so in particular the boundary has orbifold simple normal crossings.

Then $\tilde{\mathcal{A}}$ is the union of the collection of cones $\mathcal{F}(\partial\overline{Y})$. In particular, $\tilde{\mathcal{A}}$ is the underlying set of a fan, and the Bieri-Groves set $\mathcal{A}$ is the underlying set of the fibre of the map of fans induced by $\tilde{N}_{\mathbb{Q}} \to \mathbb{Q} = \operatorname{Hom}(K^*/R^*, \mathbb{Q})$.

Proof. Let $\mathcal{G} \subset \tilde{N}_{\mathbb{Q}}$ be the union of cones in the statement. Since $R^* \subset \mathcal{O}^*(\overline{Y})$ and $R \subset \mathcal{O}(\overline{Y})$, divisors of $\overline{Y}$ have trivial valuation on R^* and nonnegative valuation on R. Therefore, $[D] \in \tilde{\mathcal{A}}$. Moreover, let $\sigma = \{\mathcal{D}_1, \dots, \mathcal{D}_k\} \subset \partial\overline{\mathcal{Y}}$ with $\cap D_i \neq \emptyset$. Then any linear combination $\sum n_i[D_i]$ with nonnegative rational coefficients belongs to $\tilde{\mathcal{A}}$. Indeed, the exceptional divisors of weighted blowups of the regular sequence σ give arbitrary nonnegative combinations (up to scalar multiples). Thus $\mathcal{G} \subset \tilde{\mathcal{A}}$.

Let $S \subset L$ be the valuation ring for L, v as in Definition 2.1. As $v|_R$ is nonnegative, we have the morphism $\operatorname{Spec}(S) \to \operatorname{Spec}(R)$, and so by properness there is a unique extension $f : \operatorname{Spec}(S) \to \overline{Y}$. Let σ be the collection of all boundary divisors of $\overline{Y}$ that contain the image of the closed point of $\operatorname{Spec}(S)$. Let $u \in \mathcal{O}^*(Y)$ be a unit, and let $B \subset \partial\overline{Y}$ be its locus of poles, the union of boundary divisors with negative valuation on u. Then $u \in \mathcal{O}(\overline{Y} \setminus B)$, so if $[D](u) \geq 0$ for all $D \in \sigma$, $f^{-1}(B) = \emptyset$ so $f^*(u) \in S$ and $[v, f](u) = v(f^*(u)) \geq 0$. Thus $[v, f] \in \mathcal{F}(\sigma)$. □

2.4. COROLLARY. *Let $\mathcal{Z} \subset \overline{\mathcal{Z}}$ be an open substack of a Deligne-Mumford stack proper and smooth over* $\operatorname{Spec} k$*, with simple normal crossing boundary, and $Z \subset \overline{Z}$ the corresponding orbifold simple normal crossing compactification of coarse moduli spaces. Then $\mathcal{A}(Z)$ is the underlying set of a fan, and is the union of the collection of cones $\mathcal{F}(\partial\overline{Z})$.*

Proof. If $\mathcal{Y} = \mathcal{Z} \times_k K$ for a k-variety $\mathcal{Y}'$ then $\overline{\mathcal{Y}} := \overline{\mathcal{Z}} \times_k R$ satisfies all conditions of Theorem 2.3. It follows that $\tilde{\mathcal{A}}_Y$ is the product $\mathcal{A}_Z \times \mathbb{Q}_{\geq 0}$. The statement follows easily. □

In what follows we frequently use

2.5. THEOREM ([T, 3.1]). *A dominant map of very affine varieties $Y' \to Y$ induces a surjective homomorphism of intrinsic tori $T_{Y'} \to T_Y$ and a surjective map of tropical varieties $\mathcal{A}(Y') \to \mathcal{A}(Y)$.*

Note if Y is Hübsch, and the log canonical fan is strictly convex, then $\mathcal{A}(Y') \to \mathcal{A}(Y)$ is a map of fans, for any fan structure on $\mathcal{A}(Y')$ (and the log canonical fan structure on $\mathcal{A}(Y)$).

Proof of (1.6.1). Since discrete valuations lift to field extensions it is clear that the set defined in (1.6.1) satisfies the analog of Th. 2.5. Now by de Jong's theorem [J, 4.1] we may assume Y is smooth and admits a normal crossing compactification. Now the result follows from the proofs of Th. 2.3 and Cor. 2.4. □

Proof of Theorem 1.10. Let $k = \dim Y$. Recall that top-dimensional cones of $\mathcal{F}$ are k-dimensional. By [T, 2.5], any refinement of a tropical fan is tropical, so it is enough to show that an arbitrary fan $\mathcal{F}''$ with $|\mathcal{F}''| = \mathcal{A}_Y$ refines the log canonical fan $\mathcal{F}$. Suppose on the contrary that $\mathcal{F}''$ does not refine $\mathcal{F}$. Then there exists a $(k-1)$-dimensional cone $\alpha \in \mathcal{F}$ which meets the interior of a k-dimensional cone $\sigma'' \in \mathcal{F}''$. Let $\mathcal{F}'$ be a strictly simplicial common refinement of $\mathcal{F}$ and $\mathcal{F}''$ (with the same support). There exists a $(k-1)$-dimensional cone $\alpha' \in \mathcal{F}'$, $\alpha' \subset \alpha \cap \sigma''$, and meeting the interiors of each. Let $X' \xrightarrow{p} X''$, $X' \xrightarrow{\pi} X$ be the corresponding proper birational toric morphisms. Let Z, Z', Z'' be closed toric strata corresponding to α, α', σ''. Then

- $Z'' = T''$ is a quotient torus of T.
- $p : Z' \to T''$ is a surjective proper toric map, thus $Z' = \mathbb{P}^1 \times T''$.
- $\pi : Z' \to Z$ is birational.

By [T, 1.4], the structure map $\overline{Y}' \times T \to X'$ is smooth and surjective, and therefore $G = \overline{Y}' \cap Z' \subset \mathbb{P}^1 \times T''$ is 1-dimensional, smooth, reduced, and proper. Since G is proper and T'' is affine, it follows that $G = \mathbb{P}^1 \times z$ for a reduced 0-dimensional subscheme $z \subset T''$. So $K_G + B_G$ is trivial. G is also a closed stratum of $\overline{Y}'$ and thus (since $\overline{Y}'$ has smooth structure map) $(K_{\overline{Y}'} + B)|_G = K_G + B_G$ is trivial. Thus G is contracted by π, since $\pi : \overline{Y}' \to \overline{Y}$ is the log canonical model, and log crepant by [T, 1.4]. By equivariance of the map $\pi : Z' \to Z$, it contracts all fibres of $Z' \to Z''$, i.e., it is not birational, a contradiction. □

Suppose that $\overline{Y}$ is a compactification of the smooth variety Y with simple normal crossing boundary $\partial\overline{Y}$. We assume until the end of this section that $\mathrm{Pic}(\overline{Y})$ is a lattice generated by the classes of boundary divisors. Let $A := \mathbb{Z}^{\partial\overline{Y}}$ be the free Abelian group with a basis given by irreducible boundary divisors. We have a canonical exact sequence $0 \to \mathrm{Pic}(\overline{Y})^\vee \to A \to N_Y \to 0$, where N_Y is dual to $M_Y = \mathcal{O}^*(Y)/k^*$. Let T_{Pic}, $\mathbb{G}_m^{\partial\overline{Y}}$, T_Y be the algebraic tori with characters $\mathrm{Pic}(\overline{Y})$, $A^\vee$, M_Y. We have an exact sequence $\{e\} \to T_{\mathrm{Pic}} \to \mathbb{G}_m^{\partial\overline{Y}} \to T_Y \to \{e\}$.

Let $\mathbb{A}^{\partial\overline{Y}}$ be the affine space and let $\mathcal{U} \subset \mathbb{A}^{\partial\overline{Y}}$ be the open $\mathbb{G}_m^{\partial\overline{Y}}$-toric subvariety defined as follows: a collection of coordinates vanish simultaneously at some point of $\mathcal{U}$ iff the intersection of the corresponding divisors of $\overline{Y}$ is non-empty. Let $\tilde{\mathcal{F}} \subset A_{\mathbb{Q}}$ be the fan of $\mathcal{U}$, i.e., for each open stratum $S \subset \overline{Y}$, we take the convex hull $\tilde{\sigma}_S$ of the rays associated with each of the boundary divisors that contains S. Let $\mathcal{U}_S \subset \mathcal{U}$ be the open subset associated to $\tilde{\sigma}_S$ and $Z_S \subset \mathcal{U}_S$ the closed orbit.

2.6. Lemma. *The following conditions are equivalent:*

1. T_{Pic} *acts freely on* $\mathcal{U}_S$.
2. $\tilde{\sigma}_S$ *maps isomorphically onto a strictly simplicial cone in* N_Y.
3. *Boundary divisors that don't contain* S *generate* $\mathrm{Pic}(\overline{Y})$.
4. *For each boundary divisor* $D \supset S$*, we can find a unit* $u \in \mathcal{O}^*(Y)$ *with valuation one on* D*, and valuation zero on other boundary divisors containing* S.

Proof. Immediate from the definitions. □

2.7. Proposition. *If the (equivalent) conditions of Lemma 2.6 hold for all strata* S*, then* T_{Pic} *acts freely on* $\mathcal{U}$*, with quotient a smooth (not necessarily separated)* T_Y*-toric variety, the union of affine toric varieties* $X(\sigma_S)$ *where* $\sigma_S \subset N_Y$ *is the (isomorphic) image of* $\tilde{\sigma}_S \subset A_{\mathbb{Q}}$*. Denote this toric variety by* $X(\mathcal{F})$.

Proof. $X(\mathcal{F})$ can be glued from affine charts $X(\sigma_S) = \mathcal{U}_S/T_{\mathrm{Pic}}$ as in the standard definition of a toric variety [O]. □

Consider the sheaf of algebras $B := \bigoplus_{D\in \mathrm{Pic}(\overline{Y})} \mathcal{O}_{\overline{Y}}(D)$ with multiplication given by tensor product (more precisely, to define the multiplication we have to fix line bundles whose classes form a basis of $\mathrm{Pic}(\overline{Y})$). Let $W := \mathrm{Spec}(B) \xrightarrow{q} \overline{Y}$.

2.8. Theorem. *T_{Pic} acts freely on W with quotient $q : W \to \overline{Y}$. There is a natural T_{Pic} equivariant map $f : W \to \mathcal{U}$. Suppose that T_{Pic} acts freely on $\mathcal{U}$ with quotient $X(\mathcal{F})$ as in Prop. 2.7. Then f induces a natural map of quotients $\overline{Y} \to X(\mathcal{F})$. The scheme-theoretic inverse image of a toric stratum is a stratum of $\overline{Y}$, and this establishes a one-to-one correspondence between toric strata of $X(\mathcal{F})$ and boundary strata of $\overline{Y}$. The map $\overline{Y} \times T_Y \to X(\mathcal{F})$ is smooth and surjective.*

Proof. We can check the first statement locally on $\overline{Y}$ and so may assume that all line bundles are trivial. Then $W = \overline{Y} \times T_{\mathrm{Pic}}$ and the statement is obvious.

The sections $1_D \in H^0(\overline{Y}, \mathcal{O}(D))$ induce an equivariant map $f : W \to \mathbb{A}^{\partial\overline{Y}}$ and a scheme-theoretic equality $f^{-1}(Z_S) = q^{-1}(S)$. In particular, $f(W) \subset \mathcal{U}$. Next we show that $W \times \mathbb{G}_m^{\partial\overline{Y}} \to \mathcal{U}$ is smooth and surjective. Since both spaces are smooth, it's enough to check that all fibres are smooth (and non-empty) of the same dimension. If we pullback the map to the torus orbit Z_S we obtain

$$F : q^{-1}(S) \times \mathbb{G}_m^{\partial\overline{Y}} \to Z_S.$$

Since $\mathbb{G}_m^{\partial\overline{Y}} \to Z_S$ is a surjective homomorphism, dF is everywhere surjective. It follows that F is a smooth surjection, of relative dimension independent of S.

The statements for $\overline{Y}$ follow by taking the quotient by the free action of T_{Pic}. □

2.9. Definition. Let $M_Y^S \subset M_Y$ be the sublattice generated by units with zero valuation on all boundary divisors containing S. Note we have a canonical restriction map $M_Y^S \to M_S = \mathcal{O}^*(S)/k^*$ and thus a canonical map

$$S \to T_Y^S := \mathrm{Hom}(M_Y^S, \mathbb{G}_m). \tag{2.9.1}$$

2.10. Theorem. *Suppose that the conditions of Lemma 2.6 hold for all strata S. The map $\overline{Y} \to X(\mathcal{F})$ of Th. 2.8 is an immersion iff* (2.9.1) *is an immersion for any S.*

If S is very affine and $M_Y^S \to M_S$ is surjective then (2.9.1) *is an immersion.*

If (2.9.1) *is an immersion for any S and the image cones $\sigma_S \subset N_Y$ form a fan $\mathcal{F}$, then $X(\mathcal{F})$ is the associated toric variety, Y is Schön, and $\mathcal{F}$ is tropical.*

Proof. $\overline{Y} \to X(\mathcal{F})$ is an immersion iff each fiber is a single (reduced) point. We can check this stratum by stratum on $X(\mathcal{F})$. But one checks immediately from the definitions that the maps in the statement are exactly the maps over the various torus orbits of $X(\mathcal{F})$. If M_Y^S generates the units and S is very affine, it is clear that the map for this stratum is a closed embedding.

Finally, if the cones σ_S form a fan, it is clear that $X(\mathcal{F})$ is the associated toric variety. Since the structure map is smooth by Th. 2.8, the fan is tropical. □

§3. Iitaka fibration for subvarieties of algebraic tori

If $X \subset A$ is a smooth subvariety of an abelian variety then either K_X is ample or X is preserved by a positive dimensional subgroup $A' \subset A$. See [M, 3.7]. Here we prove an analogous statement for subvarieties of algebraic tori. This answers affirmatively a question posed to the second author by Miles Reid.

3.1. Theorem. *Let $Y \subset T$ be a Schön closed subvariety of an algebraic torus. Exactly one of the following holds:*

(1) *Y is log minimal, or*
(2) *Y is preserved by a nontrivial subtorus $S \subset T$.*

Proof. Let X be a smooth T-toric variety such that the closure $\overline{Y}$ of Y in X is smooth, projective, and transverse to the toric boundary. So $\overline{Y}$ is smooth and the restriction B of the toric boundary is a reduced simple normal crossing divisor.

By [T, 1.4], the line bundle $K_{\overline{Y}} + B$ is globally generated. So Y is log minimal iff $K_{\overline{Y}} + B$ is positive on any curve $C \subset \overline{Y}$ which is not contained in the boundary.

Suppose that Y is not log minimal, and let C be a curve as above such that $(K_{\overline{Y}} + B).C = 0$. Then C intersects B (because T is affine and so cannot contain a proper curve). In particular, $l := -K_{\overline{Y}} \cdot C = B \cdot C > 0$. Using translation by an element of T, we can assume that C contains the unit element $e \in T$ and that C is smooth at e.

By the bend-and-break argument, we can assume that C is a rational curve and $l \leq \dim Y + 1$ (use [Ko3, II.5.14], with $M := K_{\overline{Y}} + B$). Let $\nu : \mathbb{P}^1 \to C$ be the normalization.

3.2. Claim. $l > 1$.

Proof. Suppose $l = B \cdot C = 1$. Then $\nu^{-1}(B)$ is a singleton and therefore its complement is $\mathbb{A}^1$. But there are no non-constant maps $\mathbb{A}^1 \to T$. □

We choose C as above of minimal degree with respect to some polarisation of $\overline{Y}$. Let $P = \nu^{-1}(e)$. By the bend-and-break argument [W, 1.11], ν can be included in an irreducible l-dimensional family Z of morphisms $\mathbb{P}^1 \to \overline{Y}$ such that $P \mapsto e$, and the locus $\overline{S} \subset \overline{Y}$ of points spanned by these rational curves has dimension $l-1$ (Wiśniewski assumes that the class of C generates an extremal ray, but we only need that the curve C has minimal degree, see, e.g., [Ko3, IV.2.6]).

3.3. Claim. *$S := \overline{S} \cap Y$ is an algebraic subtorus of T.*

Proof. Let C' be a rational curve from the family Z. Let $\nu' : \mathbb{P}^1 \to C'$ be the normalization. Let $C'_0 := \mathbb{P}^1 \setminus \nu'^{-1}(B)$. Then $B \cdot C' = -K_X \cdot C' = l$, and therefore $\nu'^{-1}(B)$ (set-theoretically) is a union of at most l points. It follows that the intrinsic torus $T' = \operatorname{Hom}(\mathcal{O}^*(C'_0)/k^*, \mathbb{G}_m)$ has dimension at most $l-1$. By the universal property of the intrinsic torus, the morphism $C'_0 \xrightarrow{\nu'} C' \cap Y \hookrightarrow T$ factors through T'. It follows that $C' \cap Y$ is contained in a subtorus of dimension at most $l-1$. Since this is true for any curve in the family Z, and since subtori of an algebraic torus do not deform, it follows that the locus S spanned by them is contained in a subtorus of dimension at most $l-1$. But S itself is $(l-1)$-dimensional. Therefore S is an algebraic subtorus. □

We claim that Y is preserved by S. It suffices to prove that Y is preserved by any fixed one-parameter subgroup $R \subset S$. The factorization morphism $T \to T/R$ extends to the equivariant $\mathbb{P}^1$-bundle $\pi : A \to T/R$, where the fan $\mathcal{F}$ of A consists of the ray spanned by R and the opposite ray. Let $\overline{R} \simeq \mathbb{P}^1 \subset A$ be the closure of R. Since $R \subset Y$, $\mathcal{F}$ belongs to the tropicalization $\mathcal{T}(Y)$ of Y. It follows from [T, 2.5] that the closure $\tilde{Y}$ of Y in A is smooth with normal crossing boundary $\tilde{B}$. Note that $\tilde{Y}$ is not complete and that $\overline{R} \subset \tilde{Y}$. We have a morphism $f : W \to \overline{S}$ where W is the total space of a $\mathbb{P}^1$-bundle $p : W \to V$ (the universal family of smooth rational curves), such that f contracts a section of p to $e \in \overline{S}$, f is finite over $\overline{S} \setminus \{e\}$, and $f^*(K_{\overline{Y}} + B)$ is zero on fibers of p. It follows that $K_{\overline{Y}} + B$ is numerically trivial on $\overline{S}$. By [T, 1.4], the morphism $\pi : \tilde{Y} \to \overline{Y}$ is log crepant, i.e.,

$K_{\tilde{Y}} + \tilde{B} = \pi^*(K_{\overline{Y}} + B)$. Hence $(K_{\tilde{Y}} + \tilde{B})|_{\overline{R}} = 0$. Since the log canonical line bundle of $\overline{R}$ is trivial, it follows that the determinant of the normal bundle $\mathcal{N}_{\overline{R}/\tilde{Y}}$ is trivial by adjunction. The bundle $\mathcal{N}_{\overline{R}/\tilde{Y}}$ is a subbundle of $\mathcal{N}_{\overline{R}/A}$, which is trivial because it is the normal bundle of a fiber of a fibration. So $\mathcal{N}_{\overline{R}/\tilde{Y}} \simeq \oplus_{i=1}^k \mathcal{O}(a_i)$ with $a_i \leq 0$ for each i and $\sum a_i = 0$, i.e., $\mathcal{N}_{\overline{R}/\tilde{Y}}$ is the trivial bundle. Therefore, by standard deformation theory, $\tilde{Y}$ is covered by deformations of $\overline{R}$. It follows that Y is preserved by R. □

§4. Genus 3 curves: Moduli stack $\tilde{\mathcal{M}}_3^{(2)}$

For this section we assume char $k \neq 2$. We construct a Deligne–Mumford stack $\tilde{\mathcal{M}}_3^{(2)}$ that, as we will later prove, has $\overline{Y}^7_{\text{lc}}$ as its coarse moduli space. Throughout we use caligraphic font to indicate stacks, and ordinary font for coarse moduli spaces.

Let Γ be the dual graph of a stable curve, with each vertex v labeled by $g(v)$, the genus of the corresponding connected component of the normalization. Define

$$\overline{\mathcal{M}}_\Gamma = \prod_{v \in \Gamma^0} \overline{\mathcal{M}}_{g(v), E_v}$$

where E_v is the set of incident edges locally about v (so a loop counts twice).

4.1. Theorem ([GP, Appendix]). *The closed strata of $\overline{\mathcal{M}}_g$ are in bijection with dual graphs of stable curves.* $\mathrm{Aut}(\Gamma)$ *acts naturally on $\overline{\mathcal{M}}_\Gamma$ and $\overline{\mathcal{M}}_\Gamma / \mathrm{Aut}(\Gamma)$ is the normalization of the stratum corresponding to Γ.*

4.2. Lemma. *The closed stratum corresponding to the dual graph Γ fails to be smooth iff there is a dual graph A and two distinct sets of edges N_1, N_2 of A such that the curve corresponding to Γ can be obtained from the curve corresponding to A by smoothing nodes corresponding to N_1 (resp. corresponding to N_2).*

Proof. We consider the versal deformation space of the singularities. It is smooth, with one variable for each node, and coordinate hyperplanes, each corresponding to smoothing all but a single node. To each coordinate subspace (intersection of coordinate hyperplanes) is attached a dual graph, and two subspaces are local analytic branches of the same global stratum iff they have the same dual graph. Inclusion of strata corresponds to smoothing of nodes. Now the result is clear. □

4.3. Strata of $\overline{\mathcal{M}}_3$ are listed in [Fa, pg. 340–347]. We draw some of them in Fig. 1.

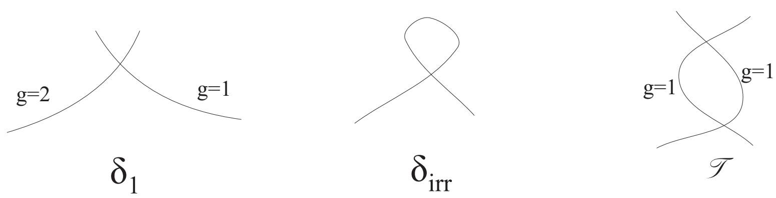


Figure 1. Some Strata of $\overline{\mathcal{M}}_3$

Strata δ_1 and δ_{irr} have codimension 1 and $\mathcal{T}$ has codimension 2. These strata are closed. Checking the criterion from Lemma 4.2 for each substratum of $\mathcal{T}$ gives

4.4. Proposition. *The substack $\mathcal{T}$ is smooth and isomorphic to*

$$(\overline{\mathcal{M}}_{1,2} \times \overline{\mathcal{M}}_{1,2})/(\mathbb{Z}/2\mathbb{Z})^2,$$

where one generator of $(\mathbb{Z}/2\mathbb{Z})^2$ interchanges the two copies of $\overline{\mathcal{M}}_{1,2}$ and the other acts diagonally on the two copies, interchanging the two marked points.

4.5. Definition. Let $\mathcal{H}\subset\mathcal{M}_3$ be the hyperelliptic divisor with closure $\overline{\mathcal{H}}\subset\overline{\mathcal{M}}_3$. By the boundary of $\overline{\mathcal{M}}_3$ we mean the complement to $\mathcal{M}_3\setminus\mathcal{H}$ (the Deligne–Mumford boundary δ_1, δ_{irr} plus $\overline{\mathcal{H}}$). Let $\tilde{\mathcal{M}}_3$ be the blowup of $\overline{\mathcal{M}}_3$ along $\mathcal{T}$. For a divisor $\mathcal{D}$ on $\overline{\mathcal{M}}_3$ we write $\tilde{\mathcal{D}}$ for its strict transform on $\tilde{\mathcal{M}}_3$. Let $\mathcal{E}$ be the exceptional divisor.

4.6. Proposition. *$(\tilde{\mathcal{M}}_3,\mathcal{B})$ is smooth with normal crossings. $\tilde{\mathcal{H}}\to\overline{\mathcal{H}}$ is an isomorphism. The normalization of $\tilde{\Delta}_1$ is $\overline{M}_{1,1}\times\overline{M}_{0,7}/S_6$.*

Proof. $\overline{\mathcal{H}}\subset\overline{\mathcal{M}}_3$ is a smooth substack with coarse moduli space isomorphic to $\overline{M}_{0,8}/S_8$. Let $\mathcal{B}_i$ ($i=2,3,4$) be the boundary divisor of $\overline{\mathcal{H}}$ that corresponds to stable rational curves with two components and with i marked points on one of the components. By [CH], δ_1 meets $\overline{\mathcal{H}}$ transversally in $\mathcal{B}_3$; $\mathcal{T}$ is contained in $\overline{\mathcal{H}}$ and equal to the substack $\mathcal{B}_4$; $\delta_{\rm irr}$ meets $\overline{\mathcal{H}}$ transversally along $\mathcal{B}_2$, but $\delta_{\rm irr}$ has two transverse analytic branches along $\mathcal{T}$, each of which meets $\overline{\mathcal{H}}$ transversally. In terms of Weil divisors $\delta_{\rm irr}|_{\overline{\mathcal{H}}}=\mathcal{B}_2+2\mathcal{B}_4$. It follows that $(\tilde{\mathcal{M}}_3,\mathcal{B})$ is smooth with normal crossings and $\tilde{\mathcal{H}}\simeq\overline{\mathcal{H}}$.

The normalization of δ_1 is $\overline{\mathcal{M}}_{1,1}\times\overline{\mathcal{M}}_{2,1}$ by Th. 4.1. Since $\mathcal{T}$ is transverse to δ_1, the normalization of $\tilde{\delta}_1$ will be the blowup of the inverse image of $\mathcal{T}$ on the normalization of δ_1. This inverse image is the product of $\overline{\mathcal{M}}_{1,1}$ with the codimension two stratum S of $\overline{\mathcal{M}}_{2,1}$ corresponding to a pointed rational curve meeting a genus 1 curve in two points. By [R, §10] the coarse moduli space of the blowup of $\overline{\mathcal{M}}_{2,1}$ along S is $\overline{M}_{0,7}/S_6$. □

4.7. Definition. Consider the locally constant sheaf $R^1_{et}c_*(\mathbb{Z}/2\mathbb{Z})$, where $c:\mathcal{C}\to\mathcal{M}_3$ is the universal curve. The associated locally constant sheaf of symplectic bases gives a Galois étale cover $\mathcal{M}_3^{(2)}\to\mathcal{M}_3$ with group $G=\mathrm{Sp}(2,6)$ [DM, §5]. For any $\mathcal{X}$ birational to $\mathcal{M}_3$ we write $\mathcal{X}^{(2)}$ for the normal closure of $\mathcal{X}$ in the function field of $\mathcal{M}_3^{(2)}$, and for any closed substack $\mathcal{Z}\subset\mathcal{X}$ we write $\mathcal{Z}^{(2)}$ for its inverse image in $\mathcal{X}^{(2)}$.

4.8. Theorem. *The pair $(\tilde{\mathcal{M}}_3^{(2)},\mathcal{B})$ is smooth with simple normal crossing boundary, in particular, irreducible components of boundary divisors are smooth. They are in $W(E_7)$-equivariant bijection with certain elements of $\mathcal{R}(E_7)$:*

Θ	A_1	A_2	$A_3\times A_3$	A_7
type of $\mathcal{D}(\Theta)$	$\tilde{\delta}^{(2)}_{\rm irr}$	$\tilde{\delta}^{(2)}_1$	$\tilde{\mathcal{E}}^{(2)}$	$\tilde{\mathcal{H}}^{(2)}$
$D(\Theta)$ *is isomorphic to*		$\overline{M}_{0,4}\times\overline{M}_{0,7}$	$\overline{M}_{0,5}\times\overline{M}_{0,5}\times\overline{M}_{0,4}$	$\overline{M}_{0,8}$

4.9. Remark. Let us recall how $\mathrm{Sp}(2,6)$ is related to E_7. Let Pic_n be the Picard group of the blow-up of $\mathbb{P}^2$ in n points with the intersection pairing $h\cdot h=1$, $e_i\cdot e_j=-\delta_{i,j}$, $h\cdot e_i=0$. Let $K_n=-3h+e_1+\ldots+e_n$ be the canonical class. Then $E_n:=\{\alpha\in(K_n)^\perp\mid\alpha^2=-2\}$. For example, positive roots of E_7 are given by

$$\alpha_{ij}=e_i-e_j;\quad \alpha_{ijk}=h-e_i-e_j-e_k;\quad \beta_i=2h-e_1-\ldots-\hat{e}_i-\ldots-e_7. \quad (4.9.1)$$

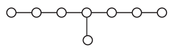
Sometimes we skip α and β and denote roots simply as ij, ijk, and i.

For any $v\in\mathrm{Pic}_7$, let $\bar{v}=v\otimes 1\in\mathrm{Pic}_7\otimes\mathbb{F}_2$. Then $(\bar{K}_7)^2=0$ and the induced bilinear form on $V:=(\bar{K}_7)^\perp/(\bar{K}_7)\simeq\mathbb{F}_2^6$ is nondegenerate and symplectic. We have

$$V\setminus\{0\}=\{\bar{\alpha}\,|\,\alpha\in E_7^+\}.$$

The Weyl group $W(E_7)$ is generated by reflections $x\mapsto x+(\alpha,x)\alpha$ for $\alpha\in E_7$. The induced transformation of V is a symplectic transvection $x\mapsto x+(\bar{\alpha},x)\bar{\alpha}$. The corresponding homomorphism $W(E_7)\to G$ is surjective with kernel $\{\pm E\}$.

4.10. Remark. We frequently use well-known facts about conjugacy classes of root subsystems in an irreducible root system Δ. Most of them can be summarized in the following observation of Eugene Dynkin: any irreducible root subsystem of Δ is $W(\Delta)$-equivalent to a subsystem such that its Dynkin diagram is contained in the affine Dynkin diagram of Δ. For example, the affine Dynkin diagram of E_7 is



It is easy to see that all subsystems of E_7 of types A_1, A_2, $A_3 \times A_3$, A_7 (vertices of $\mathcal{R}(E_7)$) are conjugate. Moreover, the action of $W(E_7)$ on $\mathcal{A}_7(E_7)$ is 2-transitive and the intersection of any 2 A_7's is $A_3 \times A_3$. Notice also that $A_2^\perp$ is isomorphic to A_5 but not all A_5's are conjugate: there is another $W(E_7)$-orbit.

Proof of Th. 4.8. Let $(\overline{\mathcal{M}}, \mathcal{B})$ be the formal neighborhood of a point on $\tilde{\mathcal{M}}_3$ or $\overline{\mathcal{M}}_3$, let $\mathcal{M} := \overline{\mathcal{M}} \setminus \mathcal{B}$. By [GM], the tame algebraic fundamental group of $\mathcal{M}$ is $\hat{\mathbb{Z}}^m$, with one generator e_i for each component of $\mathcal{B}$. The cover $\pi : \overline{\mathcal{M}}^{(2)} \to \overline{\mathcal{M}}$ gives rise to a homomorphism $h : \mathbb{Z}^m \to G$ that factors through a product of cyclic groups $\tilde{h} : \Gamma_1 \times \ldots \Gamma_m \to G$ (here we take each $|\Gamma_i|$ minimal). We use a variation of Abhyankar's lemma:

4.11. Lemma ([GM]). *The cover $\pi : \overline{\mathcal{M}}^{(2)} \to \overline{\mathcal{M}}$ is a disconnected union of $|G|/|\Gamma|$ generalized Kummer coverings, where Γ is the image of h. More precisely, π is isomorphic to the restriction of $\coprod(\mathbb{A}', B') \to (\mathbb{A}, B)$ to the germ of $(\mathbb{A}, B)$ at the origin, where $\mathbb{A} = \mathbb{A}^m$ (resp. $\mathbb{A}'$) is the toric variety associated to the first octant in the lattice $\mathbb{Z}^m$ (resp. the intersection of the octant with* Ker h*). $(\overline{\mathcal{M}}^{(2)}, \mathcal{B}^{(2)})$ is smooth with normal crossings if and only if $\tilde{h}$ is an isomorphism. In this case, if $\mathcal{D} \subset \overline{\mathcal{M}}$ is a smooth divisor transverse to the boundary, then $\pi^{-1}(\mathcal{D})$ is smooth.*

The monodromy around δ_1 and δ_{irr} is well-known, see e.g. [GO]. The analytic picture is as follows: a small disc transverse to a boundary divisor corresponds to a degeneration in which a smooth curve C degenerates to a singular curve C'. The special fibre is a deformation retract of the total family (over the disc) and this induces a surjective map on homology $H_1(C, \mathbb{Z}) \to H_1(C', \mathbb{Z})$ the kernel of which is generated by one class, the so called vanishing cycle. We use the same term for the generator of $H^1(C', \mathbb{Z})^\perp \subset H^1(C, \mathbb{Z})$ and the analogous classes for other coefficient rings, and refer to [GO, 2.10] for an algebraic description of vanishing cycles. The corresponding monodromy action on $V = H^1_{et}(C, \mathbb{Z}/2\mathbb{Z})$ is by symplectic transvection by the vanishing cycle α.

The monodromy action is trivial for loops around δ_1, as the vanishing cycle is trivial. The monodromy around $\mathcal{E}$ is a composition of monodromies around two branches of δ_{irr} meeting at $\mathcal{T}$. A generic point of $\mathcal{T}$ is a union of two elliptic curves glued at two points and the vanishing cycles for these two nodes are non-trivial but equal modulo 2. It follows that the monodromy around $\mathcal{E}$ is also trivial. Thus $\overline{\mathcal{M}}_3^{(2)} \to \overline{\mathcal{M}}_3$ (resp. $\tilde{\mathcal{M}}_3^{(2)} \to \tilde{\mathcal{M}}_3$) is étale outside of δ_{irr} (resp. $\tilde{\delta}_{\text{irr}}$).

Consider the formal neighborhood in $\tilde{\mathcal{M}}_3^{(2)}$ as in Lemma 4.11. We claim that $\tilde{h}$ is an isomorphism and therefore $(\tilde{\mathcal{M}}_3^{(2)}, \mathcal{B})$ is smooth with normal crossings. By Prop. 4.4, $\mathcal{T}$ has no self-intersection. It follows that strata contained in $\mathcal{T}$ are characterized by the existence of a unique pair of nodes which, when simultaneously removed, disconnect the curve into two connected curves. The equivalent condition on the vanishing cycles is that they are non-trivial, but equal modulo 2. Call this class α if it exists. Note that such a stratum corresponds to two disjoint strata in $\tilde{\mathcal{M}}_3$ (that both belong to $\mathcal{E}$), with the same local monodromy generated by transvections by α, and the other non-trivial vanishing cycles (for the nodes of type $\delta_{\rm irr}$ other than the distinguished pair). We get a collection of different and pairwise orthogonal vanishing cycles. Since their number is less than 7 (because $(\tilde{\mathcal{M}}_3, \mathcal{B})$ has normal crossings), the following lemma implies that $\tilde{h}$ is an isomorphism.

4.12. Lemma. *Let $S \subset V \setminus \{0\}$ be a set of pairwise orthogonal vectors. Let $H \subset G$ be the subgroup generated by transvections by elements of S. Then $H = (\mathbb{Z}/2\mathbb{Z})^{|S|}$ unless $S \cup \{0\}$ is the maximal isotropic subspace of $2^3 = 8$ vectors.*

More generally, a subset $S \subset V \setminus \{0\}$ is a reduction of a root subsystem of E_7 modulo 2 if and only if it is closed under transvections by its elements. In this case the group generated by these transvections is isomorphic to the Weyl group of the root subsystem or to its quotient modulo $\{\pm E\}$ if the latter is contained in it.

Proof. The second statement follows from the simple fact that a subset of a root system is a root subsystem if and only if it is closed under reflections by its elements.

For the first statement, S is obviously closed under transvections. The corresponding root subsystem is $A_1^{|S|}$ because any other subsystem contains nonperpendicular roots. The corresponding Weyl group is $(\mathbb{Z}/2\mathbb{Z})^{|S|}$. It is obvious that a group generated by reflections in perpendicular vectors in $\mathbb{R}^7$ contains $-E$ if and only if these vectors form a basis, i.e., there are 7 of them. □

It remains to prove that the boundary divisors are smooth (so that the boundary has *simple* normal crossings), to identify them, and to enumerate them by elements of $\mathcal{R}(E_7)$.

4.13. Lemma. *Irreducible components of $\tilde{\delta}_1^{(2)}$ are smooth and in one-to-one G-equivariant correspondence with symplectic 2-planes $W \subset V$. The latter G-set is canonically identified with $\mathcal{A}_2(E_7)$. The Galois group of the branched cover $\pi : \tilde{\Delta}_{1,W} \to \mathrm{Norm}(\tilde{\Delta}_1)$ is isomorphic to $S_3 \times S_6$ (we write $\mathrm{Norm}(X)$ for the normalization of the reduced space). The branched cover π is the natural quotient map*

$$\overline{M}_{0,4} \times \overline{M}_{0,7} \to \overline{M}_{0,4}/S_3 \times \overline{M}_{0,7}/S_6$$

Proof. We begin by identifying the Galois group — which by the general theory is the subgroup of $\mathrm{Sp}(2,6)$ that preserves a given irreducible component of $\delta_1^{(2)}$. We note that the local system $R^1\pi_*(\mathbb{Z}/2\mathbb{Z})$ is canonically split over the open stratum δ_1^0, corresponding to the splitting of the cohomology of a stable curve with a single node of type δ_1, $V = W \oplus W^\perp$ where W is the cohomology of the genus 1 component. It follows that the stabilizer is contained in the subgroup preserving W, which is equal to $\mathrm{Sp}(2,W) \times \mathrm{Sp}(2,W^\perp)$. Notice that W is nondegenerate and the set of subspaces of given signature forms a single $\mathrm{Sp}(2,6)$-orbit by the Witt theorem. Any subspace is obviously closed under transvections by its elements and therefore corresponds to a root subsystem by Lemma 4.12. It easy to see that this subsystem is A_2. Therefore the set of 2-dimensional symplectic subspaces $W \subset \mathbb{F}_2^6$ is $\mathrm{Sp}(2,6)$-equivariantly identified with $\mathcal{A}_2(E_7)$ and $\mathrm{Sp}(W)$ is equal to $W(A_2) = S_3$ (note that the symplectic group is generated by symplectic transvections). By a similar argument $W^\perp$ is identified with $A_2^\perp = A_5$ and $\mathrm{Sp}(W^\perp) = W(A_5) = S_6$.

Therefore $\mathrm{Sp}(2,W) \times \mathrm{Sp}(2,W^\perp)$ is isomorphic to $S_3 \times S_6$ and is generated by transvections by classes in either W or $W^\perp$. Such transvections are given by monodromy around boundary divisors of δ_1, corresponding to further degenerations of the curve where we shrink to a point a loop on either the elliptic, or genus 2 component. Thus it follows that the Galois group is precisely $S_3 \times S_6$. It is clear from this discussion that the cover of δ_1 is (at least generically) the product of the covers $\overline{\mathcal{M}}_{1,1}^{(2)} \to \overline{\mathcal{M}}_{1,1}$ and $\overline{\mathcal{M}}_{2,1}^{(2)} \to \overline{\mathcal{M}}_{2,1}$. By [DO, VIII.3] and Prop. 4.6, after pulling back to $\tilde{\mathcal{M}}_3$ and passing to the coarse moduli spaces, these covers are the quotients

$\overline{M}_{0,4} \to \overline{M}_{0,4}/S_3$ and $\overline{M}_{0,7} \to \overline{M}_{0,7}/S_6$. So it is enough to prove that components of $\tilde{\delta}^{(2)}_{\mathrm{irr}}$ are normal.

Since they are Cohen-Macaulay (because the ambient stack is smooth) it's enough to check there is no self intersection in codimension one. Such a self intersection would lie over a codimension one self intersection of δ_1. There is a unique such stratum in $\overline{\mathcal{M}}_3$, $\Delta_{1,1}$ of [Fa, pg. 340]. This corresponds to a curve C with two nodes, each of type δ_1. It's clear that smoothing the two nodes gives rise to distinct and orthogonal $W_1, W_2 \subset H^1(C, \mathbb{Z}/2\mathbb{Z})$, and thus, on the cover, local analytic branches that belong to distinct irreducible components. □

4.14. Lemma. *Irreducible components of $\overline{H}^{(2)}$ are smooth, isomorphic to $\overline{M}_{0,8}$ and in one-to-one G-equivariant correspondence with quadratic forms Q of plus type inducing the symplectic form of $\mathbb{F}_2^6$. This G-set is canonically identified with $\mathcal{A}_7(E_7)$. Each component $\overline{\mathcal{H}}_Q$ is isomorphic to its strict transform on $\tilde{\mathcal{M}}_3^{(2)}$. Irreducible components of $\tilde{\mathcal{H}}^{(2)}$ are pairwise disjoint. The Galois group of the cover $\overline{H}_Q \to \overline{H}$ is the subgroup of G preserving Q. It is isomorphic to S_8 naturally acting on $\overline{M}_{0,8}$.*

Proof. A nonsingular quadratic form on $\mathbb{F}_2^{2m}$ has *plus type* if there exists a totally isotropic subspace of dimension m [A, p. xi]. The symplectic group acts transitively on the quadratic forms of plus type inducing the symplectic form. The parametrization of irreducible components of $\overline{H}^{(2)}$ by quadratic forms of plus type and description of the Galois group is given in [DO, VIII.3], where in addition the cover $\overline{H}_Q \to \overline{H}$ is generically identified with $M_{0,8} \to M_{0,8}/S_8$.

Irreducible components of $\tilde{\mathcal{H}}^{(2)}$ are smooth by (4.11) since $\tilde{\mathcal{H}}$ is smooth and transverse to the branch locus of $\pi : \tilde{\mathcal{M}}_3^{(2)} \to \tilde{\mathcal{M}}_3$. In particular they are normal, so, since $\overline{H} = \overline{M}_{0,8}/S_8$, the cover $\overline{H}_Q \to \overline{H}$ is the quotient $\overline{M}_{0,8} \to \overline{M}_{0,8}/S_8$. To show that each component of $\tilde{\mathcal{H}}^{(2)}$ maps isomorphically onto its image in $\overline{\mathcal{M}}_3^{(2)}$ it suffices to show that each component of $\overline{\mathcal{H}}^{(2)}$ is normal. It is enough to check singular points of the cover, in codimension one over $\overline{\mathcal{H}}$, and there is only one such stratum, $\mathcal{T}$, of $\overline{\mathcal{M}}_3$. A generic point of $\mathcal{T}$ is a union of two elliptic curves glued at two points and the vanishing cycles for these two nodes are non-trivial but equal modulo 2. It follows by Th. 4.11 that the cover $\overline{\mathcal{M}}_3^{(2)} \to \overline{\mathcal{M}}_3$ is given locally by (the product with $\mathbb{A}^4$ of) $(z^2 = xy) \subset \mathbb{A}^3 \to \mathbb{A}^2$ by dropping the z coordinate. Here $x = 0$, $y = 0$ are the two analytic branches of δ_{irr}, each of which meets $\overline{\mathcal{H}}$ transversally along $\mathcal{T}$. Thus the inverse image of $\overline{\mathcal{H}}$ has two smooth analytic branches meeting transversally. It is enough to argue that the two branches belong to different irreducible components of $\overline{\mathcal{H}}^{(2)}$. Fix one and let Q be the corresponding quadratic form. Let α be the vanishing cycle in V (for either of the two nodes). We compute $Q(\alpha)$.

Consider a smooth hyperelliptic curve degenerating to a curve of type $\mathcal{T}$. We obtain this by taking two genus 1 curves, and on each choosing a disc and its image under a hyperelliptic involution, and now joining the two curves, to obtain a hyperelliptic curve with two thin collars, interchanged by the involution. In the degeneration the collars contract to a pair of conjugate points. $\mathbb{F}_2^6 = H^1(C, \mathbb{Z}/2\mathbb{Z})$ is identified with even cardinality subsets of the branch locus, with addition exclusive or, modulo identifying a subset with its complement. Q assigns to a set half its cardinality, modulo 2, see [DO, VIII.3]. The symplectic form assigns to a pair of subsets the cardinality of their intersection, modulo 2. Under this identification, the collar circle α is represented by the sum of the 4 branch points on one of the two elliptic curves. Thus $Q(\alpha) = 0$. Transvection by α interchanges the two sheets of the local analytic model of the cover of $\overline{\mathcal{M}}_3$ above. It does not preserve Q: Let

A be the subset representing α and Z a pair of points with $|A \cap Z| = 1$. One checks that $Q(Z) = 1$ but $Q(Z + (Z, A)A) = 0$. Thus the other analytic branch of the inverse image of $\overline{\mathcal{H}}$ belongs to a different irreducible component.

It remains to construct an $A_7 \subset E_7$ using the form Q. Recall the description of the root system E_7 from (4.9). As above, we identify $\mathbb{F}_2^6$ with even cardinality subsets of the set $\{1, \ldots, 8\}$ modulo identifying a subset with its complement. Then $\bar{\alpha}_{ij}$ corresponds to $\{i, j\}$, $\bar{\alpha}_{ijk}$ corresponds to $\{1, \ldots, 7\} \setminus \{i, j, k\}$, and $\bar{\beta}_{i8}$ corresponds to $\{1, \ldots, 7\} \setminus \{i\}$. The set $\{Q(x) = 1\}$ then corresponds to a subsystem

$$\{\alpha_{ij},\ \beta_{i8}\}. \tag{4.14.1}$$

It is easy to see that this is a root subsystem of type A_7. □

There is a unique $W(E_7)$-orbit of root systems of type A_1^7 in E_7. We call such a root system a *Fano simplex* (because it corresponds to the set of (-2)-curves on the surface over a field k of characteristic 2 obtained by blowing up the seven points of the Fano plane $\mathbb{P}^2(\mathbb{F}_2)$ in $\mathbb{P}^2_k$).

4.15. Lemma. *Irreducible components of $\tilde{\delta}^{(2)}_{\text{irr}} \subset \tilde{\mathcal{M}}^{(2)}_3$ are smooth. The G-set of irreducible components is canonically identified with $\mathcal{A}_1(E_7) \simeq V \setminus \{0\}$. A collection of $\tilde{\delta}_{\text{irr},\alpha}$ divisors has non-trivial intersection iff the roots $\alpha \in E_7$ are pairwise orthogonal and don't form a Fano simplex.*

Proof. Let $W \subset \overline{\mathcal{M}}_3$ be the complement of all the codimension two boundary strata. To compute the set I of irreducible components of $\delta^{(2)}_{\text{irr}}$ we may replace $\overline{\mathcal{M}}_3$ by the formal completion U of W along $D := \delta_{\text{irr}} \cap W$. Then $I = G/H$ for $H \subset G$ the image of the monodromy map $\pi_1^D(U) \to G$. We have an exact sequence [GM, 7.3]

$$\{e\} \to K \to \pi_1^D(U) \to \pi_1(D) \to \{e\},$$

with K central and cyclic, generated by monodromy around δ_{irr} (this is the algebraic analog of the fibration sequence for the unit sphere bundle in the normal bundle to a submanifold). The image of K in H is generated by transvection by a vanishing cycle α. As K is central, $H \subset G_\alpha$. It is easy to check by direct calculation that G_α is generated by transvections by elements in $\alpha^\perp$, and as in the proof of Lemma 4.13 we can realize such a transvection by monodromy in U (by considering degenerations contracting two disjoint cycles to points). Thus $H = G_\alpha$. So as a G-set

$$I = G/G_\alpha = V \setminus \{0\} = \mathcal{A}_1(E_7).$$

Now we argue that each component $\tilde{\delta}_\alpha$ of $\tilde{\delta}^{(2)}_{\text{irr}}$ is smooth. As in the proof of (4.13) we consider a divisor of self intersection. There are two such divisors for $\delta_{\text{irr}} \subset \overline{\mathcal{M}}_3$, one of which, $\mathcal{T}$, is removed by the blowup. So a self intersection divisor of $\tilde{\delta}_\alpha$ must lie over the strict transform of the stratum $\Delta_{0,0}$ of [Fa, pg. 340]. This corresponds to a curve with two δ_{irr} nodes, for which there are two distinct vanishing cycles. The cover is smooth over this stratum, and $\tilde{\delta}^{(2)}$ has two analytic braches. They belong to the two irreducible components corresponding to the two vanishing cycles.

Finally an intersection point of k δ_{α_i} divisors lies over a k-fold self intersection of δ_{irr}. Such strata are listed in [Fa, pg 340–347]. Clearly the collection of k vanishing cycles are pairwise orthogonal. In the other direction, it's easy to check that any two k element subsets of pairwise orthogonal elements of V are conjugate under $\mathrm{Sp}(2, V)$ unless $k = 3$ or 4 in which case there are two conjugacy classes. It follows that all such intersections occur. □

4.16. Lemma. *Irreducible components of $T^{(2)}$ (resp $E^{(2)}$) are smooth, pairwise disjoint, and isomorphic to $\overline{M}_{0,5} \times \overline{M}_{0,5}$ (resp. $\overline{M}_{0,5} \times \overline{M}_{0,5} \times \overline{M}_{0,4}$). They are in one-to-one correspondence with root subsystems of type $A_3 \times A_3$.*

Proof. The description of $T^{(2)}$ follows from Lemma 4.14 since $T \subset \overline{H} = \overline{M}_{0,8}/S_8$ is the boundary divisor B_4. The correspondence with root systems of type $A_3 \times A_3$ now follows from (4.10), where it is noted that $A_3 \times A_3 \subset E_7$ are in one to one correspondence with pairs $A_7 \neq A_7' \subset E_7$, under $A_7 \cap A_7' = A_3 \times A_3$. The local analytic description of the cover $\overline{\mathcal{M}}_3^{(2)} \to \overline{\mathcal{M}}_3$ in the proof of Lemma 4.14 shows the components of $T^{(2)}$ are pairwise disjoint.

In order to describe $E^{(2)}$, we need to analyze the stacky structure at $\mathcal{T}^{(2)} \subset \overline{\mathcal{M}}_3^{(2)}$. Recall that the substack $\mathcal{T} \subset \overline{\mathcal{M}}_3$ is the intersection of two branches of δ_{irr}, and that $\overline{\mathcal{H}} \subset \overline{\mathcal{M}}_3$ also contains $\mathcal{T}$ and intersects each branch of δ_{irr} transversely. Locally over a point of $\mathcal{T}$, a connected component of the cover $\overline{\mathcal{M}}_3^{(2)} \to \overline{\mathcal{M}}_3$ is the double cover branched over δ_{irr}. Thus on $\overline{\mathcal{M}}_3^{(2)}$, a slice transverse to $\mathcal{T}^{(2)}$ is an A_1-singularity, and each component of $\mathcal{T}^{(2)}$ is contained in two branches of $\delta_{\text{irr}}^{(2)}$ and two branches of $\overline{\mathcal{H}}^{(2)}$, any two of which intersect tranversely (cutting out $\mathcal{T}^{(2)}$). The exceptional divisor $\mathcal{E}^{(2)}$ of the blowup $\tilde{\mathcal{M}}_3^{(2)} \to \overline{\mathcal{M}}_3^{(2)}$ is a $\mathbb{P}^1$-bundle over $\mathcal{T}^{(2)}$, and on each component the strict transforms of the branches of δ_{irr} and $\overline{\mathcal{H}}$ give 4 disjoint sections.

The irreducible components of the stack $\overline{\mathcal{H}}^{(2)}$ have coarse moduli space $\overline{M}_{0,8}$. For a smooth hyperelliptic curve C, the hyperelliptic involution acts trivially on $H^1(C, \mathbb{Z}/2\mathbb{Z})$, so preserves a given 2-level structure. It follows that $\overline{\mathcal{H}}^{(2)}$ has stabiliser $\mathbb{Z}/2\mathbb{Z}$ at a general point given by the hyperelliptic involution. Since $\overline{\mathcal{H}}^{(2)}$ and its coarse moduli space are both smooth, the automorphism group of an arbitrary point is generated by the specialisations of automorphisms in codimension one. We find that the automorphisms are generated by the hyperelliptic involution and the involution along $\mathcal{B}_3$ given by the hyperelliptic involution of the elliptic component, which is the restriction of the involution along δ_1.

The components of $\mathcal{T}^{(2)}$ are the closed substacks of $\overline{\mathcal{H}}^{(2)}$ with coarse moduli spaces the boundary divisors of type $\overline{M}_{0,5} \times \overline{M}_{0,5}$ in $\overline{M}_{0,8}$. The components of $\mathcal{E}^{(2)}$ are $\mathbb{P}^1$-bundles over these stacks. Note that the stabiliser $\mathbb{Z}/2\mathbb{Z}$ of a general point of $\mathcal{T}^{(2)}$ acts nontrivially on the fibre of the bundle: it fixes the sections given by $\tilde{\mathcal{H}}^{(2)}$ pointwise and interchanges the sections given by $\tilde{\delta}_{\text{irr}}^{(2)}$ (because the hyperelliptic involution of a general point $[C]$ of $\mathcal{T}$ interchanges the two nodes of C). The remaining generators of the stabiliser of an arbitrary point of $\mathcal{T}^{(2)}$ are given by the involution along $\delta_1^{(2)}$ so act trivially on the fibre. Hence a component of the coarse moduli space $E^{(2)}$ is a $\mathbb{P}^1$-bundle over $\overline{M}_{0,5} \times \overline{M}_{0,5}$ with 3 disjoint sections given by one component of $\tilde{\delta}_{\text{irr}}^{(2)}$ and two components of $\tilde{\mathcal{H}}^{(2)}$. It follows that the bundle is trivial, so each component of $E^{(2)}$ is isomorphic to $\overline{M}_{0,5} \times \overline{M}_{0,5} \times \overline{M}_{0,4}$, as required. □

§5. Irreducible representation $M(\Delta)$ of the Weyl group

5.1. Definition. Let $\Delta = \{\alpha \in \Lambda \mid \alpha^2 = -2\}$ be an A-D-E root system, where Λ is a negative-definite $\mathbb{Z}$-lattice spanned by Δ. Consider the linear map

$$\phi : \operatorname{Sym}^2 \Lambda^\vee \to \mathbb{Z}^{\mathcal{A}_1(\Delta)}, \quad f \mapsto \sum_{A_1 \in \mathcal{A}_1(\Delta)} f(A_1)[A_1],$$

where $\operatorname{Sym}_2(\Lambda^\vee)$ is the space of quadratic forms on Λ and $f(A_1)$ is equal to the value of f on one of the two (opposite) roots of A_1. We define $N(\Delta) := \operatorname{Coker} \phi$.

We choose positive roots $\Delta_+ \subset \Delta$ and identify $\mathcal{A}_1(\Delta)$ with Δ_+. Let $\{\alpha_i\}$ be the associated simple roots, which we identify with vertices of the Dynkin diagram Γ. Recall that any root $\alpha \in \Delta_+$ is a nonnegative linear combination $\sum n_i\alpha_i$ of simple roots and the support $\operatorname{Supp} \alpha = \{\alpha_i \mid n_i \neq 0\} \subset \Gamma$ is a connected subgraph.

5.2. Proposition. *We have an exact W-equivariant sequence of free $\mathbb{Z}$-modules*

$$0 \to \operatorname{Sym}^2 \Lambda^\vee \xrightarrow{\phi} \mathbb{Z}^{\Delta_+} \xrightarrow{\psi} N(\Delta) \to 0. \tag{5.2.1}$$

Let $M(\Delta)$ be the dual lattice with the induced embedding $\psi^\vee : M(\Delta) \hookrightarrow \mathbb{Z}^{\Delta_+}$. We have (1.13.1). Let $\Delta_+^T \subset \Delta_+$ be the set of roots with 3-legged support. Then $\operatorname{rk} N(\Delta) = |\Delta_+^T|$ and the restrictions $n_\alpha|_{\operatorname{Im}\psi^\vee}$ for $\alpha \in \Delta_+^T$ are coordinates on $M(\Delta)$.

Proof. Since Γ is a tree, there exists a (not unique) total ordering $\prec$ on the set of pairs of vertices $S = \{(i,j) \mid 1 \le i \le j \le \operatorname{rk}\Delta\}$ that satisfies the following property. If $(i,j) \prec (i',j')$ then either i' or j' is not contained in the minimal substring (a tree with two legs) of Γ that contains both i and j. We denote this substring by $[i,j]$. It is well-known that $\alpha_{i,j} := \sum\limits_{k\in[i,j]} \alpha_k$ is a root of Δ, moreover, $\alpha_{i,j}$ is the unique root with support $[i,j]$. Let $\{\omega_i\}$ be the basis of $\Lambda^\vee$ dual to $\{\alpha_i\}$. It follows from the property of the ordering and the definition of $\alpha_{i,j}$ that

$$\omega_i\omega_j(\alpha_{i',j'}) = \begin{cases} 1 & (i',j') = (i,j) \\ 0 & (i',j') \prec (i,j) \end{cases}$$

It follows that ϕ is injective and that $N(\Delta)$ is torsion-free. Since the $\alpha_{i,j}$'s are the only roots in $\Delta_+ \setminus \Delta_+^T$, $\operatorname{rk} N(\Delta) = |\Delta_+^T|$. Finally, (1.13.1) follows by dualizing (5.2.1). □

5.3. Corollary. *$M(\Delta)$ is an irreducible W-module of rank given in the table*

Δ	A_n	D_n	E_6	E_7	E_8
$\operatorname{rk} M(\Delta) = \lvert\Delta_+^T\rvert$	0	$\frac{n(n-3)}{2}$	15	35	84

Proof. Irreducibility follows from the tables of characters in [A]. □

5.4. Lemma. *If $\Delta' \subset \Delta$ then we have a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Sym}^2\Lambda^\vee(\Delta) & \longrightarrow & \mathbb{Z}^{\Delta_+} & \longrightarrow & N(\Delta) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Sym}^2\Lambda^\vee(\Delta') & \longrightarrow & \mathbb{Z}^{\Delta'_+} & \xrightarrow{\psi} & N(\Delta') & \longrightarrow & 0 \end{array} \tag{5.4.1}$$

where the first two vertical arrows are restrictions of functions. The induced map $N(\Delta) \to N(\Delta')$ is surjective. In particular, $M(\Delta') \to M(\Delta)$ is injective.

Proof. Clear from definitions. □

5.5. Lemma. *Consider the standard partial order on Δ_+: $\alpha \ge \beta$ iff $\alpha - \beta = \sum n_i\alpha_i$, where $n_i \ge 0$. If $\alpha > \beta$ then there exists a sequence of simple roots $\alpha_{i_1}, \ldots, \alpha_{i_p}$ such that $\alpha = \beta + \alpha_{i_1} + \ldots + \alpha_{i_p}$ and $\beta + \alpha_{i_1} + \ldots + \alpha_{i_k} \in \Delta_+$ for any $k = 1, \ldots, p$.*

Proof. Let $\alpha - \beta = \sum_{i\in I} n_i\alpha_i$, where $n_i > 0$ for any $i \in I$. Recall that the inner product on Λ is negative-definite. Arguing by induction on $\sum n_i$, it suffices to prove that there exists $i \in I$ such that $\beta + \alpha_i$ or $\alpha - \alpha_i$ is a root. Suppose that the index with this property does not exist. Then $(\beta, \alpha_i) \le 0$ and $(\alpha, \alpha_i) \ge 0$ for any $i \in I$. Therefore, $(\alpha - \beta, \alpha - \beta) = (\alpha - \beta, \sum n_i\alpha_i) \ge 0$. This is a contradiction. □

5.6. Lemma. *There exists a unique decomposition $D_4^+ = F_1 \coprod F_2 \coprod F_3$, where each F_i is a fourtuple of pairwise orthogonal roots. $M(D_4) \subset \mathbb{Z}^{D_4^+}$ consists of functions that are constant on each fourtuple F_i and add up to 0.*

Proof. By (5.2), $M(D_4) \subset \mathbb{Z}^{\mathcal{A}_1(D_4)}$ consists of linear combinations

$$\sum_{1 \le i < j \le 4} a_{ij}[e_i - e_j] + b_{ij}[e_i + e_j]$$

such that

$$\sum_{1 \le i < j \le 4} a_{ij}(e_i - e_j)^2 + b_{ij}(e_i + e_j)^2 = 0.$$

Looking at the coefficient at $e_i e_j$, we see that $a_{ij} = b_{ij}$ and the remaining condition is that $\sum\limits_{1 \le i < j \le 4} a_{ij}(e_i^2 + e_j^2) = 0$. A short calculation gives the claim. □

5.7. THEOREM. *The map* $\bigoplus\limits_{D_4 \subset \Delta} M(D_4) \to M(\Delta)$ *induced by* (5.4.1) *is surjective. In particular, the map* $N(\Delta) \to \bigoplus\limits_{D_4 \subset \Delta} N(D_4)$ *is injective.*

Proof. We will prove by induction on the partial order $\ge$ of Lemma 5.5 that

5.8. CLAIM. *For any* $\alpha \in \Delta_+^T$ *there exists a* D_4 *such that* $D_4^+ = F_1 \coprod F_2 \coprod F_3$ *as in Lemma 5.6,* $\alpha \in F_1$*, and all other roots of* F_1 *and* F_2 *are smaller than* α*.*

Assuming the Claim, let $I \subset M(\Delta)$ be the image of $\bigoplus\limits_{D_4 \subset \Delta} M(D_4)$. By the Claim, for any $\alpha \in \Delta_+^T$, I contains a function $\alpha + \ldots$, where all other terms are less than α. Using induction and Prop. 5.2, we conclude that $I = M(\Delta)$.

Now we prove the Claim. We again argue by induction on $\ge$. Let $\mathbb{D}_4 \subset \Delta$ be the unique D_4 with $\Gamma(D_4) \subset \Gamma(\Delta)$. Let $\alpha_0 \in \mathbb{D}_4$ be the sum of the simple roots. α_0 is the smallest root in Δ_+^T and one checks that α_0 is greater than other positive roots of $\mathbb{D}_4$ except for one root, which we include in F_3. Thus the Claim holds for α_0.

Let $\alpha \in \Delta_+^T$. Since $\alpha \ge \alpha_0$, by Lemma 5.5 there exists a simple root γ such that $\alpha - \gamma$ is a root and $\alpha - \gamma \ge \alpha_0$, i.e., $\alpha - \gamma \in \Delta_+^T$. By the induction hypothesis, the Claim is true for $\alpha - \gamma$. Let $D_4^+ = F_1 \coprod F_2 \coprod F_3$ be the corresponding D_4. Let $r \in W(\Delta)$ be the reflection w.r.t. γ. Then $r(\alpha - \gamma) = \alpha$. Consider $D_4' = r(D_4)$ and the fourtuples F_i' obtained from $r(F_i)$ by replacing any negative root with its opposite. Then $\alpha \in F_1'$ and we claim that any other root $\pm r(\delta)$ in F_1' or F_2' is smaller than α. Indeed, if $\delta \ne \gamma$ then $r(\delta) \in \Delta_+$ (because r permutes roots in $\Delta_+ \setminus \{\gamma\}$) and is equal to $\delta + \gamma$, δ, or $\delta - \gamma$. In any case, $r(\delta) < \alpha$ because $\delta < \alpha - \gamma$. If $\delta = \gamma$ then $r(\delta) = -\gamma$ and the corresponding positive root in a fourtuple is γ. And $\gamma < \alpha$. □

5.9. REMARK ([S2]). We will need the list of all possible $D_4 \subset E_7$. For any partition $\{ijkl \,|\, a \,|\, uv\}$, $\{ij \,|\, kl \,|\, mn \,|\, b\}$, or $\{ab \,|\, cd \,|\, ijk\}$ of $\{1 \ldots 7\}$, let $D(ijkl, a)$, $D(ij, kl, mn)$, or $D(ab, cd)$ respectively, be the D_4-subsystem with fourtuples

$$D(ijkl, a) = \{\{ij, kl, aij, akl\}; \quad \{ik, jl, aik, ajl\}; \quad \{il, jk, ail, ajk\}\}$$

$$D(ij, kl, mn) = \{\{ij, kl, mn, b\}; \quad \{ikm, iln, jkn, jlm\}; \quad \{ikn, ilm, jkm, jln\}\}$$

$$D(ab, cd) = \{\{i, jk, abi, cdi\}; \quad \{j, ik, abj, cdj\}; \quad \{k, ij, abk, cdk\}\}.$$

5.10. THEOREM. *We define* $\psi(\Theta)$ *as in* (1.14)*. Let* $\Theta = D_k \subset D_n$*. Then* $\psi(\Theta) = \psi(\Theta^\perp) = 2\psi(A_{k-1})$ *for any* $A_{k-1} \subset D_k$*. In Table 1 we list conjugacy classes of irreducible root subsystems in* E_6 *and* E_7 *and the relation between their* ψ*-images.*

In a few cases $\psi(\Theta)$ *is divisible in the lattice* $N(\Delta)$*:* $\frac{1}{2}\psi(D_k) \in N(D_n)$*,* $\frac{1}{4}\psi(D_k \times D_k) \in N(D_{2k})$*,* $\frac{1}{3}\psi(A_2^{\times 3}) \in N(E_6)$*,* $\frac{1}{2}\psi(A_3^{\times 2}) \in N(E_7)$*,* $\frac{1}{4}\psi(A_7) \in N(E_7)$*.*

Θ	$\Theta^\perp \subset E_6$	ψ	f
A_1	A_5	$\psi(A_5) = 3\psi(A_1)$	$\sum_{i=1}^{6} x_i(d+x_i)$
A_2	$A_2^{\times 2}$	$2\psi(A_2) = \psi(A_2^{\times 2})$	$\sum_{1\le i<j\le 3} (x_ix_j + x_{i+3}x_{j+3}) - d^2$
A_3	$A_1^{\times 2}$	$\psi(A_3) = \psi(A_1^{\times 2})$	$x_1x_2 - x_3x_4 + x_5x_6 - d(x_3+x_4) - d^2$
A_4	A_1	$\psi(A_4) = 2\psi(A_1)$	$x_1(d+x_1) - \sum_{i=2}^{6} x_i(d+x_i)$
A_5	A_1		
D_4	$\emptyset$	$\psi(D_4) = 0$	$\sum_{i=2}^{5} x_i^2 - x_6^2 - (d+x_1)(x_1+3d+2x_6)$
D_5	$\emptyset$	$\psi(D_5) = 0$	$3d^2 + x_1^2 + 4x_1d - \sum_{i=2}^{6} x_i^2$

Θ	$\Theta^\perp \subset E_7$	ψ	f
A_1	D_6	$\psi(D_6) = 3\psi(A_1)$	$3d^2 + x_1^2 + 4x_1d - \sum_{i=2}^{7} x_i^2$
A_2	A_5^-	$\psi(A_5^-) = 2\psi(A_2)$	$3d^2 + 2d(x_1+x_2) - 2x_1x_2 + \sum_{i=1}^{2} x_i^2 - \sum_{i=3}^{7} x_i^2$
A_3	$A_3' \times A_1$	$\psi(A_3) = \psi(A_3')$	$\sum_{i=1}^{3} x_i(d+x_i) - \sum_{i=4}^{7} x_i(d+x_i)$
A_5^-	A_2		
D_4	$A_1^{\times 3}$	$\psi(D_4) = \psi(A_1^{\times 3})$	$\sum_{i=2}^{5} x_i^2 - \sum_{i=6}^{7} x_i^2 - (d+x_1)(x_1+3d+2\sum_{i=6}^{7} x_i)$
D_5	A_1	$\psi(D_5) = 2\psi(A_1)$	$3d^2 + x_1^2 + 4x_1d - \sum_{i=2}^{7} x_i^2 - 4dx_7 - 2x_7\sum_{i=2}^{6} x_i$
D_6	A_1		
E_6	$\emptyset$	$\psi(E_6) = 0$	$d^2 - \sum_{i=1}^{6} x_i^2 + x_7^2$
A_7	$\emptyset$		
A_4	A_2	$4\psi(A_4) = 4\psi(A_2) + \psi(A_7)$	$5\sum_{i=1}^{2} x_i(d+x_i) - 3\sum_{i=3}^{7} x_i(d+x_i)$
A_5^+	A_1	$2\psi(A_5^+) = 2\psi(A_1) + \psi(A_7)$	$3x_1(d+x_1) - \sum_{i=2}^{7} x_i(d+x_i)$
A_6	$\emptyset$	$4\psi(A_6) = 3\psi(A_7)$	$\sum_{i=1}^{7} x_i(d+x_i)$

TABLE 1. Root subsystems of E_6 and E_7. In the last three rows, A_7 is a unique subsystem of this type that contains both Θ and $\Theta^\perp$.

Proof. The classification of irreducible subsystems is well-known [S1, S2]. In each case it is easy to compute $\Theta^\perp$. We will check only three equalities involving $\psi(\Theta)$, leaving other cases to the reader. Our proof is a routine calculation using Prop. 5.2, which identifies elements of $\mathbb{Z}^{\Delta_+}$ trivial in $N(\Delta)$ with functions of the form $f(\alpha)$, where $f \in \operatorname{Sym}^2 \Lambda^\vee(\Delta)$. We provide f in each case in Table 1.

Let $\Theta = D_k \subset \Delta = D_n$. Write $D_I = \{\pm\varepsilon_i \pm \varepsilon_j \mid i, j \in I\}$ for $I \subset \{1, \ldots, n\}$. We can assume that $\Theta = D_{1,\ldots,k}$ and $\Theta^\perp = D_{k+1,\ldots,n}$. Let $A_{k-1} \subset D_k$ be the standard subsystem of Lemma 7.2. Let $x_1, \ldots, x_n$ be standard coordinates on $\Lambda(D_n) \subset \mathbb{Z}^n$. Consider $f = x_1^2 + \ldots + x_k^2 - x_{k+1}^2 - \ldots - x_n^2$. Then $f(\alpha) = 2$ if $\alpha \in \Theta$, $f(\alpha) = -2$ if $\alpha \in \Theta^\perp$, and $f(\alpha) = 0$ otherwise. It follows that $\psi(\Theta) = \psi(\Theta^\perp)$. Now consider $f = \sum_{1\le i<j\le k} x_ix_j$. Then $f(\alpha) = -1$ if $\alpha \in A_{k-1}$, $f(\alpha) = 1$ if $\alpha \in D_k \setminus A_{k-1}$, and $f(\alpha) = 0$ otherwise. So $\psi(D_k) = 2\psi(A_{k-1})$.

Let $\Theta = A_2 \subset \Delta = E_6$. Then $A_2^\perp = A_2' \times A_2''$, where we can assume that $A_2 = \{123, 7, 456\}$, $A_2' = \{45, 56, 46\}$, and $A_2'' = \{12, 23, 13\}$. We denote by $\{d, x_1, \ldots, x_6\}$ the standard coordinates on Pic_6 and their restrictions on $\Lambda(E_6)$.

Consider the function $f \in \operatorname{Sym}^2 \Lambda^\vee(E_6)$ from the $A_2 \subset E_6$ row of Table 1. Then $f(\alpha) = -1$ if $\alpha \in A_2' \times A_2''$, $f(\alpha) = 2$ if $\alpha \in A_2$, and $f(\alpha) = 0$ otherwise. It follows that $-\psi(A_2' \times A_2'') + 2\psi(A_2) = 0$.

Take the standard $A_7 \subset E_7$ (4.14.1). It contains an A_6 with roots α_{ij}. We use coordinates on $\Lambda(E_7)$ as above. Take $f \in \operatorname{Sym}^2 \Lambda^\vee(E_7)$ from the $A_6 \subset E_7$ row of Table 1. Then $f(\alpha) = 2$ for $\alpha \in A_6$, $f(\alpha) = -6$ for $\alpha \in A_7 \setminus A_6$, and $f(\alpha) = 0$ otherwise. It follows that $8\psi(A_6) - 6\psi(A_7) = 0$.

It remains to check divisibility of $\psi(\Theta)$. In D_n, $\psi(D_k) = 2\psi(A_{k-1})$. In D_{2k}, $\psi(D_k \times D_k) = \psi(D_k \times D_k^\perp) = 2\psi(D_k) = 4\psi(A_{k-1})$. In E_6, $\psi(A_2^{\times 3}) = \psi(A_2) + \psi(A_2^\perp) = 3\psi(A_2)$. In E_7, $\psi(A_3^{\times 2}) = \psi(A_3) + \psi(A_3') = 2\psi(A_3)$. Finally, we have $4\psi(A_6) = 3\psi(A_7)$, and therefore $\psi(A_7)$ is divisible by 4. □

§6. Intrinsic torus of $Y(\Delta)$ and KSBA cross-ratios

Now we identify $M(\Delta)$, combinatorially defined in §5, with units of $Y(\Delta)$.

6.1. Lemma. *Let $Z_1, \dots, Z_k \subset \mathbb{G}_m^l$ be distinct irreducible Weil divisors defined by equations $F_1, \dots, F_k \in \mathcal{O}(\mathbb{G}_m^l)$. Then $Y := \mathbb{G}_m^l \setminus \cup Z_i$ is very affine, with intrinsic torus $\mathbb{G}_m^{k+l}$, and $\mathcal{O}^*(Y)/k^*$ is generated by the F_i and the coordinates of $\mathbb{G}_m^l$.*

Proof. Let W be the graph of $\mathbb{G}_m^l \xrightarrow{F_1,\dots,F_k} \mathbb{A}^k$. Then $W \cap \mathbb{G}_m^{k+l}$ is closed in $\mathbb{G}_m^{k+l}$ and isomorphic to Y via the first projection. Therefore Y is very affine. Factoriality of $\mathbb{G}_m^l$ implies that units of Y have form $\chi F_1^{p_1} \dots F_k^{p_k}$, where χ is a unit of $\mathbb{G}_m^l$. □

6.2. We define $Y(D_n) := M_{0,n}$. Here $D_3 = A_3$. The Weyl group $W(D_n)$ acts on $M_{0,n}$ through its quotient S_n (permuting marked points). Fixing the first 3 points, we identify $M_{0,n}$ with an open subset of $\mathbb{A}^{n-3}$: a point $(z_1, \dots, z_{n-3}) \in \mathbb{A}^{n-3}$ corresponds to n points in $\mathbb{P}^1$ given by the columns of the matrix

$$\begin{bmatrix} 1 & 0 & 1 & z_1 & \dots & z_{n-3} \\ 0 & 1 & 1 & 1 & \dots & 1 \end{bmatrix}.$$

The boundary of $M_{0,n}$ in $\mathbb{A}^{n-3}$ consists of the hyperplanes $z_i = 0$, $z_i = 1$, and $z_i = z_j$. Therefore $M_{0,n}$ is an open subvariety of $\mathbb{G}_m^{n-3} \subset \mathbb{A}^{n-3}$ with divisorial boundary and so is very affine, with intrinsic torus of rank $\frac{n(n-3)}{2}$ by Lemma 6.1.

6.3. $Y(E_n) = Y^n$ is the moduli space of marked del Pezzo surfaces, $n = 4, 5, 6, 7, 8$. Here $E_4 := A_4$ and $E_5 := D_5$. A *marking* of S is an isometry $m : \operatorname{Pic}_n \to \operatorname{Pic} S$ such that $m(K_n) = K_S$. We denote $H = m(h)$ and $E_i = m(e_i)$. Two marked surfaces (S, m) and (S', m') are isomorphic if there exists an isomorphism $f : S \to S'$ such that $m = f^* \circ m'$. $W(E_n)$ acts on Y^n by twisting markings: $w \cdot (S, m) = (S, m \circ w)$.

Let X^n be the quotient of the open subset of $(\mathbb{P}^2)^n$ of points in linearly general position by the free action of PGL_3. As above, we identify X^n with an open subset of $\mathbb{A}^{2n-8}$ by taking n points in $\mathbb{P}^2$ given by columns of the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 1 & x_1 & \dots & x_{n-4} \\ 0 & 1 & 0 & 1 & y_1 & \dots & y_{n-4} \\ 0 & 0 & 1 & 1 & 1 & \dots & 1 \end{bmatrix}. \tag{6.3.1}$$

For any $(S, m) \in Y^n$, $\mathcal{O}_S(H)$ gives a morphism $S \to \mathbb{P}^2$ that blows down $E_1, \dots, E_n$ to n points $p_1, \dots, p_n \in \mathbb{P}^2$ such that no 3 points lie on a line, no 6 points lie on a conic, and no 8 points lie on a cubic curve singular at one of them [D]. For any collection of points with these conditions, their blow up is a del Pezzo surface [D]. Therefore, Y^n is isomorphic to an open subset of $\mathbb{A}^{2n-8}$ with a divisorial boundary [S1, §4] that includes the hyperplanes $x_i = 0$ and $y_i = 0$. Therefore Y^n is very affine by Lemma 6.1.

Blowing up the n sections (6.3.1) of the trivial bundle $Y^n \times \mathbb{P}^2$ gives a universal family of del Pezzo surfaces $\mathcal{S} \to Y^n$. The natural morphism

$$p : Y^{n+1} \to Y^n \tag{6.3.2}$$

given by blowing down E_{n+1} is induced by the projection $\mathbb{A}^{2n-6} \to \mathbb{A}^{2n-8}$ given by truncating the matrix (6.3.1). Therefore it factors through an open embedding $i : Y^{n+1} \hookrightarrow \mathcal{S}$ with a divisorial boundary that we denote by $\mathcal{B}$.

$$\begin{array}{ccc} Y^{n+1} & \hookrightarrow & (\mathcal{S}, \mathcal{B}) \\ & \searrow & \downarrow \\ & & Y^n \end{array} \tag{6.3.3}$$

The fiber of $\mathcal{B}$ over $[S] \in Y^n$ is identified with the union of (-1)-curves in S and (if $n = 7$) the ramification curve of the double cover $S \to \mathbb{P}^2$ given by $-K_S$. It is well-known (see e.g. [D]) that the number of irreducible components in $\mathcal{B}$ is equal to 10 if $n = 4$, 16 if $n = 5$, 27 if $n = 6$, and $57 = 56 + 1$ if $n = 7$.

6.4. Lemma. (6.3.3) *is $W(E_n)$-equivariant, in particular $W(E_n)$ permutes irreducible components of $\mathcal{B}_n$. In the Grothendieck ring of $W(E_n)$-modules, one has*

$$[\mathcal{O}^*(Y^{n+1})/k^*)] = [\mathcal{O}^*(Y^n)/k^*] + [\mathbb{Z}^{\mathcal{B}_n}] - [\mathrm{Pic}_n]. \tag{6.4.1}$$

In particular, $\mathrm{rk}\, \mathcal{O}^*(Y^n)/k^* = \mathrm{rk}\, M(E_n)$.

Proof. We have an exact sequence of sheaves on $\mathcal{S}^n$

$$0 \to \mathcal{O}^*_{\mathcal{S}^n} \to i_* \left[\mathcal{O}^*_{Y^{n+1}}\right] \xrightarrow{\text{val}} \bigoplus_{k=1}^{\#\mathcal{B}_n} F_k \to 0,$$

where F_k is a push-forward of the constant sheaf $\mathbb{Z}$ on the k-th irreducible component of $\mathcal{B}_n$ and val is the order of vanishing along components of $\mathcal{B}_n$. The long exact sequence of cohomology gives

$$0 \to \mathcal{O}^*(Y^n) \to \mathcal{O}^*(Y^{n+1}) \to \mathbb{Z}^{\#\mathcal{B}_n} \to \mathrm{Pic}_n \to 0,$$

which implies (6.4.1). The equality of ranks follows by counting and (6.4.1). □

6.5. Definition. Let $\mathcal{X}(\Delta)$ be the complement of the Coxeter arrangement

$$\mathcal{H} = \{(\alpha = 0) \mid \alpha \in \Delta_+\} \subset \Lambda_k^\vee.$$

We define a map $\Psi : \mathcal{X}(\Delta) \to Y(\Delta)$ as follows.

Let $\varepsilon_1, \dots, \varepsilon_n$ (7.2.1) be the coordinates on $\Lambda_k^\vee(D_n)$. Consider the map

$$\Psi : \mathcal{X}(D_n) \to M_{0,n}, \quad \Psi(\varepsilon_1, \dots, \varepsilon_n) = (\mathbb{P}^1;\ \varepsilon_1^2, \dots, \varepsilon_n^2).$$

The image is $M_{0,n}$ because $\varepsilon_i = \pm\varepsilon_j$ are precisely the root hyperplanes.

Let $q_1 = h - 3e_1, \dots, q_n = h - 3e_n$ be coordinates on $\Lambda_k^\vee(E_n)$ defined using (4.9). We define $\Psi : \mathcal{X}(E_n) \to Y^n$ as follows: $\Psi(q_1, \dots, q_n)$ is the blow up of $\mathbb{P}^2$ in points

$$p_1 = (q_1 : q_1^3 : 1), \quad \dots, \quad p_n = (q_n : q_n^3 : 1).$$

Our points lie on a cuspidal cubic curve (with a cusp at the infinity). The blow-up is a del Pezzo surface [D]: the points are distinct $\Leftrightarrow q_i \neq q_j$, no three points lie on a line $\Leftrightarrow q_i + q_j + q_k \neq 0$, no six points lie on a conic $\Leftrightarrow q_{i_1} + \dots + q_{i_6} \neq 0$, and no eight points lie on a cubic singular at one of them $\Leftrightarrow 2q_{i_1} + q_{i_2} \dots + q_{i_8} \neq 0$. One can check that these equations are precisely the root hyperplanes.

6.6. THEOREM. *There exists a $W(\Delta)$-equivariant commutative diagram*

$$\begin{array}{ccc} M(\Delta) & \xrightarrow{\psi^\vee} & \mathbb{Z}^{\Delta_+} \\ f\downarrow & & h\downarrow \\ \mathcal{O}^*(Y(\Delta))/k^* & \xrightarrow{\Psi^*} & \mathcal{O}^*(\mathcal{X}(\Delta))/k^* \end{array}$$

where h and f are isomorphisms, and $h([\alpha])$ is equal to α as a function on $\Lambda_k^\vee$.

Proof. It is clear that $\mathcal{O}^*(\mathcal{X})/k^* = \mathbb{Z}^{\Delta_+}$. Since the complement to a subarrangement given by simple roots is an algebraic torus $\mathbb{G}_m^n$, it follows from Lemma 6.1 that $\mathcal{X}$ is very affine with the intrinsic torus $T_{\mathcal{X}} = \mathrm{Hom}(\mathbb{Z}^{\Delta_+}, \mathbb{G}_m)$.

Let $\Psi^* : \mathcal{O}^*(Y(\Delta))/k^* \to \mathcal{O}^*(\mathcal{X})/k^* = \mathbb{Z}^{\Delta_+}$ be the pull-back of units. By Lemma 6.4, $\mathrm{rk}\, M(\Delta) = \mathrm{rk}\, \mathcal{O}^*(Y(\Delta))/k^*$. Therefore, it suffices to prove that any $m \in M(\Delta) \subset \mathbb{Z}^{\Delta_+}$ is a pull-back of a unit of $Y(\Delta)$. By Th. 5.7 it suffices to prove that, for any $D_4 \subset \Delta$, any $m \in M(D_4) \subset M(\Delta)$ is a pull-back of a unit. By 4.10, it suffices to prove this for any fixed $D_4 \subset \Delta$. Using the action of $W(D_4)$ and Lemma 5.6, it suffices to construct as a pull-back of a unit the function in $\mathbb{Z}^{D_4^+} \subset \mathbb{Z}^{\Delta_+}$ that is equal to 0, 1, and -1 on three fourtuples of orthogonal A_1's.

Suppose $\Delta = D_n$. Consider a forgetful map $M_{0,n} \to M_{0,4}$. A pull-back of a unit of $M_{0,4}$ (i.e. a cross-ratio of points $p_1, p_2, p_3, p_4 \in \mathbb{P}^1$) to $\mathcal{X}(D_n)$ is equal to

$$\frac{(\varepsilon_1^2 - \varepsilon_2^2)(\varepsilon_3^2 - \varepsilon_4^2)}{(\varepsilon_1^2 - \varepsilon_3^2)(\varepsilon_2^2 - \varepsilon_4^2)} = \frac{(\varepsilon_1 - \varepsilon_2)(\varepsilon_1 + \varepsilon_2)(\varepsilon_3 - \varepsilon_4)(\varepsilon_3 + \varepsilon_4)}{(\varepsilon_1 - \varepsilon_3)(\varepsilon_1 + \varepsilon_3)(\varepsilon_2 - \varepsilon_4)(\varepsilon_2 + \varepsilon_4)}.$$

In the additive notation, this function is equal to 1 on the four orthogonal roots of the numerator and to -1 on the four orthogonal roots of the denominator. By (7.2.1), these fourtuples belong to a D_4.

Suppose $\Delta = E_n$. A cross-ratio of projections of points p_1, p_2, p_3, p_4 from p_5 is a unit of $Y(\Delta)$. This is a geometric cross-ratio of type I, where (in the notation of Def. 1.4), L_5 is an exceptional divisor over p_5 and $L_1, \ldots, L_4$ are strict transforms of lines connecting $p_1, \ldots, p_4$ with p_5. The pull-back of this unit to $\mathcal{X}(E_n)$ is

$$\frac{\begin{vmatrix} q_1 & q_2 & q_5 \\ q_1^3 & q_2^3 & q_5^3 \\ 1 & 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} q_3 & q_4 & q_5 \\ q_3^3 & q_4^3 & q_5^3 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} q_1 & q_3 & q_5 \\ q_1^3 & q_3^3 & q_5^3 \\ 1 & 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} q_2 & q_4 & q_5 \\ q_2^3 & q_4^3 & q_5^3 \\ 1 & 1 & 1 \end{vmatrix}}. \tag{6.6.1}$$

Using the identity $\begin{vmatrix} a & b & c \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$, (6.6.1) is equal to

$$\frac{(q_3 - q_4)(q_1 - q_2)(q_3 + q_4 + q_5)(q_1 + q_2 + q_5)}{(q_1 - q_3)(q_2 - q_4)(q_1 + q_3 + q_5)(q_2 + q_4 + q_5)} = \frac{\alpha_{34}\alpha_{12}\alpha_{345}\alpha_{125}}{\alpha_{13}\alpha_{24}\alpha_{135}\alpha_{245}}.$$

By (5.9), these fourtuples of orthogonal roots belong to a D_4. □

6.7. COROLLARY. *For $D_4 \subset \Delta$, the composition of morphisms*

$$\mathcal{O}^*(M_{0,4})/k^* = M(D_4) \overset{i}{\hookrightarrow} M(\Delta) = \mathcal{O}^*(Y(\Delta))/k^*$$

is the pull-back of units induced either by a forgetful map $M_{0,n} \to M_{0,4}$ (if $\Delta = D_n$) or by a KSBA cross-ratio map of type I (if $\Delta = E_n$).

§7. Simplicial complex $\mathcal{R}(\Delta)$ of boundary divisors

7.1. Notation. Let $\overline{Y}(D_n) := \overline{M}_{0,n}$ and let $\overline{Y}(E_6)$ be Naruki's space of cubic surfaces. For char $k \neq 2$, let $\overline{Y}(E_7)$ be the coarse moduli space for $\tilde{\mathcal{M}}_3^{(2)}$. Note that by [DO], the coarse moduli space $\tilde{M}_3^{(2)} \setminus B$ is smooth and isomorphic to $Y(E_7)$. In what follows, whenever we refer to $\overline{Y}(E_7)$ we implicitly assume char $k \neq 2$.

7.2. Lemma. *Let $\mathbb{Z}^n$ be the standard $\mathbb{Z}$-lattice with the product $\varepsilon_i \cdot \varepsilon_j = -\delta_{ij}$. Then*

$$D_n = \{\alpha \in \mathbb{Z}^n \mid \alpha^2 = -2\} = \{\pm\varepsilon_i \pm \varepsilon_j \mid 1 \leq i < j \leq n\}. \tag{7.2.1}$$

The Weyl group $W(D_n) \simeq S_n \ltimes (\mathbb{Z}_2)^{n-1}$, where $(\mathbb{Z}_2)^{n-1}$ acts by even sign changes. Let $N_n := \{1, 2, \dots n\}$. $\mathcal{R}(D_n)$ is equivariantly isomorphic ($\mathbb{Z}_2^{n-1}$ acts trivially) to

- *the set of boundary divisors of $\overline{M}_{0,n} = \overline{Y}(D_n)$;*
- *the set of bipartitions $I \coprod I^c = N_n$, where $2 \leq |I| \leq |I^c|$;*
- *the set of subsystems of type A_k for $n > 2k+2$ and $A_l \times A_l$ if $n = 2l+2$ of the 'standard' $A_{n-1} \subset D_n$ with roots $\varepsilon_i - \varepsilon_j$, $i \neq j \in N_n$.*

Simplices are subsets of divisors with non-empty intersection, pairwise compatible bipartitions, and pairwise orthogonal or nested subsystems, respectively. Note also that if we fix $N_{n-1} \subset N_n$ then under the action of $S_{n-1} \subset S_n$, $\partial\overline{M}_{0,N_n}$ is equivariantly isomorphic to the collection of subsets of N_{n-1} of cardinality between 2 and $n-2$.

Proof. To any bipartition we assign a boundary divisor $\delta_{I,I^c} \subset \overline{M}_{0,n}$ (see e.g. [K]), a subsystem $D_I = \{\pm\varepsilon_i \pm \varepsilon_j \,|\, i,j \in I\} \subset D_n$ of type $D_{|I|}$ (if $|I| = |I^c|$ then we assign $D_I \times D_{I^c}$), and the intersection of this subsystem of D_n with A_{n-1}. □

7.3. Complex $\mathcal{R}(E_n)$. Suppose that $\sigma = \{\Theta_1, \dots, \Theta_k\} \in \mathcal{R}(E_n)$ has the property that (for some choice of simple roots) $\Gamma(\Theta_i) \subset \tilde{\Gamma}(E_n)$ for any i, where $\tilde{\Gamma}(E_n)$ is the affine Dynkin diagram. Then we can 'tube' σ as in [ARW] by drawing k 'tubes' encircling $\Gamma(\Theta_1), \dots, \Gamma(\Theta_k)$. To avoid confusion, we box reducible subsystems $A_2 \times A_2 \times A_2$ and $A_3 \times A_3$ instead of circling. We define a few simplices on Fig. 2, where VII_9 is tubed on the diagram $\Gamma'(E_7)$ obtained by adding the root α_{167} to $\tilde{\Gamma}(E_7)$.

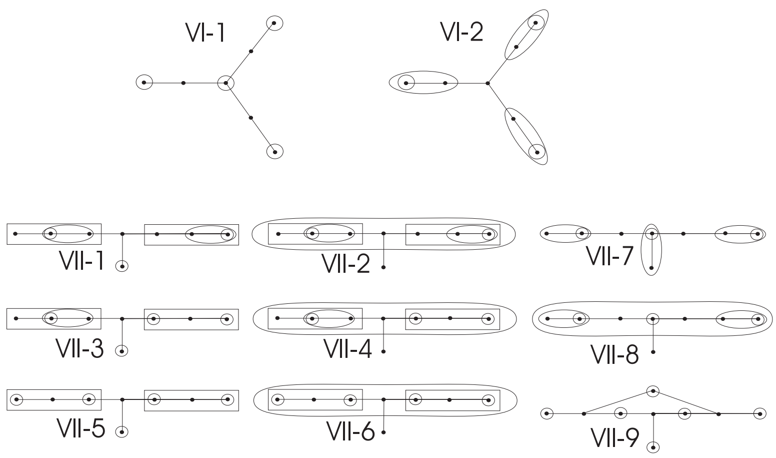


Figure 2. Maximal Simplices of $\mathcal{R}(E_6)$ and $\mathcal{R}(E_7)$

7.4. Proposition. *Points of $\mathcal{R}(E_n)$ are $W(E_n)$-equivariantly identified with boundary divisors of $\overline{Y}(E_n)$. Its simplices correspond to subsets of intersecting divisors. Any simplex of $\mathcal{R}(E_n)$ is $W(E_n)$-equivalent to a face of a simplex in Figure 2.*

Proof. The identification of vertices of $\mathcal{R}(E_6)$ with divisors of $\overline{Y}(E_6)$ and the description of maximal simplices are explicit in [N].

In Th. 4.8 we parametrized boundary divisors of $\tilde{\mathcal{M}}_3^{(2)}$ by points of $\mathcal{R}(E_7)$. We also identified $D(A_2)$, $D(A_3 \times A_3)$, and $D(A_7)$ as the moduli spaces $\overline{M}_{0,4} \times \overline{M}_{0,7}$, $\overline{M}_{0,5} \times \overline{M}_{0,5} \times \overline{M}_{0,4}$, and $\overline{M}_{0,8}$. The full boundary has simple normal crossings, and restricts to the boundary on each of these spaces. We first check that divisors $D(\Gamma)$ and $D(\Delta)$ intersect if and only if Γ and Δ are orthogonal or nested. If both the divisors have type $D(A_1)$, see Lemma 4.15. If $\Gamma = A_1$ and $\Delta = A_2$, or if $\Gamma = \Delta = A_2$, this is clear from the proof of Lemma 4.13. $A_3 \times A_3$ divisors are pairwise disjoint by Lemma 4.16.

7.5. Lemma. *If $\Delta = A_7$ then $D(\Gamma) \cap D(\Delta) \neq \emptyset$ iff $\Gamma \subset \Delta$. Subsystems A_1, A_2, $A_3 \times A_3 \subset A_7$ correspond to boundary divisors of $\overline{M}_{0,8} \simeq D(A_7)$ as in Lemma 7.2.*

Proof. A_7 divisors are pairwise disjoint by Lemma 4.14. By the proof of Lemma 4.14, $\delta_{\mathrm{irr},\alpha} \cap \mathcal{H}_Q \neq \emptyset$ iff $Q(\alpha) = 1$. And $Q(\alpha) = 1$ iff $A_1(\alpha) \subset A_7(Q)$. Now consider $\mathcal{H}_Q \cap \delta_{1,W} \neq \emptyset$. One checks using the description of Q in the proof of Lemma 4.14 that $Q|_W = 1$, or equivalently $A_2(W) \subset A_7(Q)$. The case $\Gamma = A_3 \times A_3$ follows from Lemma 4.14, where it is shown that $\mathcal{T}^{(2)}$ is the transverse intersection of two $\overline{\mathcal{H}}_Q^{(2)}$ that correspond to two A_7's intersecting in an $A_3 \times A_3$. For each type of Γ there is only one corresponding divisor of $\tilde{\mathcal{H}}$: it is clear from Lemma 4.6 that for δ_{irr} this is of type B_2, for δ_1 of type B_3, and for $\mathcal{E}$ of type B_4. □

Let $\Gamma = A_1$ or A_2 and $\Delta = A_3 \times A_3$. Let $b : \tilde{\mathcal{M}}_3^{(2)} \to \overline{\mathcal{M}}_3^{(2)}$. Then $b(D(\Delta))$ is $\mathcal{T}^{(2)}$, the self-intersection of $\delta_{irr,\alpha}$ for $\alpha = \Delta^{\perp}$. $b(D(\Gamma))$ either contains $\mathcal{T}^{(2)}$ (in which case $\Gamma = A_1 = \Delta^{\perp}$) or intersects it along a divisor. But then $\Gamma \subset \Delta$ as only cycles in Δ can be vanishing cycles for degenerations of a generic curve in $\mathcal{T}^{(2)}$.

Now for each divisor we consider the simplicial complex for its boundary. We check that these links agree with the corresponding links of $\mathcal{R}(E_7)$. For A_1 this follows from Lemma 4.15. For the remaining divisors, it follows from the well-known fact (Lemma 7.2) that a subset of divisors of $\overline{M}_{0,n}$ (and thus of products $\overline{M}_{0,n_1} \times \ldots \times \overline{M}_{0,n_k}$) has a non-empty intersection iff they intersect pairwise. □

§8. Fan of real components $\mathcal{G}(\Delta)$

8.1. Motivation. Fix $\Delta \neq E_8$. Before proving that $\mathcal{F}(\Delta)$ is a strictly simplicial fan supported on $\mathcal{A}(Y(\Delta))$, we introduce a remarkable fan $\mathcal{G}(\Delta) \subset N_{\mathbb{Q}}$ which contains $\mathcal{F}(\Delta)$ but is much more symmetric: its maximal cones are permuted by W. The origins of the fan are geometric: Y is defined over $\mathbb{R}$ and the set $Y(\mathbb{R})$ is divided into connected components, which we call cells. The Weyl group acts transitively on the set of cells[1]. The closure of each cell in $\overline{Y}(\Delta)$ is diffeomorphic (as manifold with corners) to a polytope, which we denote $P(\Delta)$. This is clear locally near the boundary (since we will prove that $\overline{Y}(\Delta)$ has normal crossings), but the additional claim is that the closure is contractible. $P(D_n)$ is identified with the associahedron in [De], and $P(E_n)$ is described in [SY, S3] (for E_7 without the proof).

Faces of $P(\Delta)$ correspond to the boundary divisors of $\overline{Y}(\Delta)$ which meet the closure of the cell (the face being the intersection of the boundary divisor with the closure of the cell). This gives a natural way of dividing up the boundary divisors, and we define the fan $\mathcal{G}(\Delta)$ by declaring (the convex hull of) a collection of rays (from $\mathcal{F} \subset N_{\mathbb{Q}}$) to be a cone if the corresponding boundary divisors all meet a

[1]This is easy for D_n and for E_n this follows from the classical result that real del Pezzo surfaces of the same topological type are connected in the moduli space, see [Ko4] for the modern exposition. Points of $Y(\mathbb{R})$ correspond to real del Pezzo surfaces that are blow-ups of $\mathbb{P}^2$ in n real points.

single closed cell, i.e., they correspond to faces of $P(\Delta)$. Since the zero strata of $\overline{Y}$ are real points, each cone of $\mathcal{F}^{\partial\overline{Y}}$ is a cone of $\mathcal{G}(\Delta)$. The remarkable fact is that the number of faces of $P(\Delta)$ is equal to the rank of $N(\Delta)$ (except in the E_7 case, where the rank is one larger). This raises the hope that such cones form a maximal dimensional, strictly simplicial fan, or equivalently that for each pair of a closed cell and one of its maximal faces, one can find a unit (necessarily unique up to scaling) which vanishes to order one along the face (i.e. along the corresponding boundary divisor) and is generically regular and non-vanishing on the other faces. This indeed turns out to be the case, and in fact the units are D_4 units. Our initial inspiration for this fan comes from [B], where the units are given in the D_n case.

We will not prove these statements and will not use this geometric description. It will suffice for our purposes to define $\mathcal{G}(\Delta)$ combinatorially.

Recall that any collection of A_1's in Δ gives rise to a graph, with vertices indexed by A_1's and where vertices are connected by an edge iff the corresponding A_1's are not orthogonal. For example, a collection of simple roots gives the Dynkin diagram, adding the lowest root gives the affine Dynkin diagram, etc.

8.2. Definition. An *n-gon diagram* is a set of A_1's in A_{n-1} which form an n-gon.

A *pentadiagram* is a collection of 10 A_1's in E_6 which form the Petersen graph.

A *tetradiagram* is a collection of 10 A_1's in E_7 which form the graph with 10 vertices given by the 4 vertices of a tetrahedron, together with the 6 midpoints of the 4 edges, with edges the 12 half edges of the tetrahedron, see Figure 3

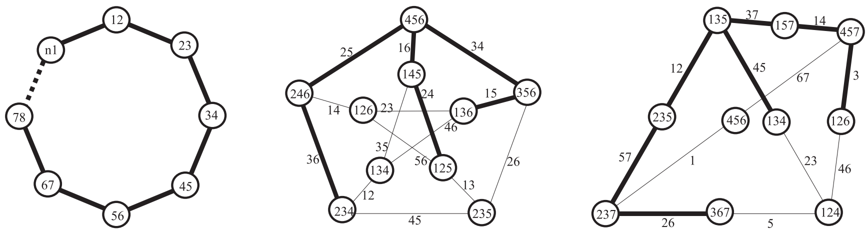


Figure 3. Sekiguchi's n-gon, pentadiagram, and tetradiagram with highlighted affine Dynkin diagrams. We mark each edge of the last two diagrams by a unique root that forms an A_2 with vertices of the edge.

8.3. Theorem (Sekiguchi). *For $\Delta = D_n$, E_6, and E_7, the Weyl group acts transitively on the set of n-gons, pentadiagrams, and tetradiagrams, respectively. The normalizer of a diagram is respectively the dihedral group D_{2n}, S_5, and $S_4 \times (\mathbb{Z}/2\mathbb{Z})$. Its action on the diagram is a natural one: the action of the dihedral group on the n-gon, the action of S_5 on the Petersen graph $\mathcal{R}(D_5)$ from Lemma 7.2, and the action of S_4 permuting vertices of the tetrahedron ($\mathbb{Z}/2\mathbb{Z}$ acts trivially).*

8.4. Lemma. *For each tetradiagram there is a unique A_1 which is perpendicular to the $A_1^{\times 6}$ formed by the midpoints of the edges of the tetrahedron.*

Proof. Uniqueness follows from $\operatorname{rk} E_7 = 7$. For Figure 3, the extra root is α_{247}. □

8.5. Definition. We define rays of $\mathcal{G}$ to be the same as rays of $\mathcal{F}$. We define a maximal cone in $\mathcal{G}$ for each n-gon, pentadiagram, and tetradiagram, respectively. We will check that they are indeed cones, in fact strictly simplicial ones, in Th. 8.7.

(D_n) We label an edge from $\varepsilon_i - \varepsilon_j$ to $\varepsilon_j - \varepsilon_k$ by j – so the diagram is now an n-gon with edges labeled by the elements of $N := \{1, \dots, n\}$. Each non-edge chord defines a bipartition $N = I \coprod I^c$, and so a ray of $\mathcal{F}(D_n)$. A collection of rays forms a cone in $\mathcal{G}(D_n)$ iff the corresponding chords do not intersect in the interior of the n-gon. The number of rays is equal to the number of chords of an n-gon, i.e., $\frac{n(n-3)}{2} = \dim N(D_n)$.

(E_6) A pentadiagram gives rise to 10 A_1's in E_6 (the vertices) and 5 subsystems of type $A_2 \times A_2 \times A_2$ — subsystems of this type whose Dynkin diagram is a subdiagram of the pentadiagram. A collection of rays forms a cone in $\mathcal{G}(E_6)$ iff they all come from a single pentadiagram. The number of rays is equal to $15 = \dim N(E_6)$.

(E_7) A tetradiagram gives rise to 10 A_1's in E_7 (the vertices), 12 A_2's (the edges), $3+6$ subsystems of type $A_3 \times A_3$ and 3 A_7's – all the subsystems of these types whose Dynkin diagrams are subdiagrams of the tetradiagram. The number of rays of the corresponding cone in $\mathcal{G}(E_7)$ is 34, i.e. $\dim N(E_7) - 1$. Let $\mathcal{G}'(E_7)$ be the fan obtained by adding to each maximal cone of $\mathcal{G}(E_7)$ the extra ray of Lemma 8.4.

8.6. Lemma. *Each cone of* $\mathcal{F}(\Delta)$ *is a cone of* $\mathcal{G}(\Delta)$.

Proof. For D_n: it is easy to check by induction that any set of compatible irreducible subsystems of A_{n-1} can be simultaneously tubed on the Dynkin diagram. In particular it can be tubed on the affine Dynkin diagram, i.e. on the n-gon.

For E_n: By Prop. 7.4, any maximal simplex of $\mathcal{R}(E_n)$ can be tubed on the affine Dynkin diagram $\tilde{\Gamma}(E_n)$ (except VII_9). But $\tilde{\Gamma}(E_n)$ is a subgraph of the pentadiagram (resp. tetradiagram), see Fig. 3. Finally, VII_9 is a simplex of 6 orthogonal A_1's which can be realized by the midpoints of the edges of the tetrahedron. □

8.7. Theorem. *Let* R *be a ray of a maximal cone* $\sigma \in \mathcal{G}$ *(*$\mathcal{G}'$ *for* E_7*) defined above. There exists a unique* D_4 *and a unique pair of* 4*-tuples* $F_1 \coprod F_2 \subset \mathcal{A}_1(D_4)$ *as in Lemma 5.6 such that* $u(F_1, F_2)$ *is equal to* 1 *on the first lattice point along* R *and is equal to* 0 *on other rays of* σ*. Here* $u(F_1, F_2) \in M(D_4)$ *is the function that is equal to* 1 *on* F_1*,* -1 *on* F_2*, and* 0 *on* F_3*. In particular,* σ *is a strictly simplicial cone.*

Proof. By symmetry we can assume that σ is given by Figure 3. Strict simpliciality follows from the previous statement, since the number of rays is equal to $\dim N$. Uniqueness follows as well. It clearly suffices to prove that $u(F_1, F_2)$ is equal to 1 on some point of R, and so by Th. 5.10, it will suffice to prove that

$$u(F_1, F_2)(\Theta) = |F_1 \cap \Theta| - |F_2 \cap \Theta| = d, \tag{8.7.1}$$

where $\Theta \in \mathcal{R}(\Delta)$ gives the ray R and $d|\psi(\Theta)$ in $N(\Delta)$ (d is given in Th. 5.10).

Proof for D_n. Here rays are given by chords R of the n-gon: induced bipartitions $N = I \coprod I^c$ correspond to subsystems $\Theta_R = D_I \times D_{I^c} \subset D_n$. Note that $\psi(\Theta_R) \in R$ and is divisible by 4 by Th. 5.10. Up to dihedral symmetry, R corresponds to the bipartition $\{1, \ldots, k\} \coprod \{k+1, \ldots, n\}$. It is illustrated in Figure 4. We take the $D_4 = D_{k,k+1,n,1}$ and fourtuples $F_1 = \{\varepsilon_1 \pm \varepsilon_k,\ \varepsilon_{k+1} \pm \varepsilon_n\}$, $F_2 = \{\varepsilon_1 \pm \varepsilon_{k+1},\ \varepsilon_n \pm \varepsilon_k\}$. Note that $F_1 \subset \Theta_R$ and $F_2 \cap \Theta_R = \emptyset$. So $u(F_1, F_2)$ is equal to 4 on $\psi(\Theta_R)$.

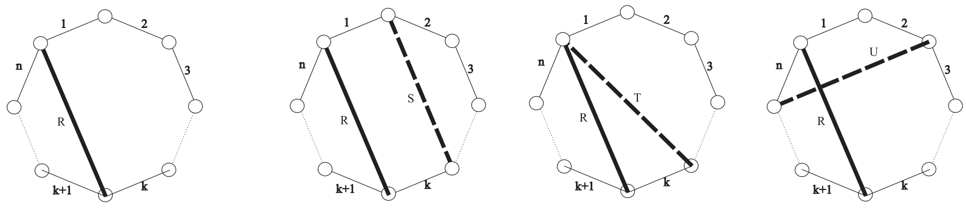


Figure 4.

It remains to prove that $u(F_1, F_2)$ vanishes on other rays of σ. There are three cases illustrated in Figure 4. If S is a chord that does not intersect R (including endpoints), then $F_1, F_2 \subset \Theta_S$. And so $u(F_1, F_2)$ is equal to $4-4=0$ on $\psi(\Theta_S)$. The next chord T intersects R at the endpoint. Then $|F_1 \cap \Theta_T| = |F_2 \cap \Theta_T| = 2$ and $u(F_1, F_2)$ is equal to $2-2=0$ on $\psi(\Theta_T)$. Finally, U is a chord that crosses R. Then $F_1 \cap \Theta_T = F_2 \cap \Theta_T = \emptyset$ and $u(F_1, F_2)$ is equal to $0-0=0$ on $\psi(\Theta_U)$.

Proof for E_6. By Th. 8.3, the subgroup of W normalizing the maximal cone of $\mathcal{G}$ is S_5. There are two S_5 orbits on rays, corresponding to the type of the subsystems, either $\Theta = A_1$ or $\Theta = A_2^{\times 3}$. Note that we can read all their roots from Figure 3.

$\Theta = A_1$. By symmetry we can assume $\Theta = 456$. D_4 is determined as follows. It is clear that $\Theta \in F_1$ and no other vertex of the pentadiagram is in F_1 or F_2. Next, Θ is contained in three subsystems Γ of type $A_2^{\times 3}$ in the pentadiagram. Each has form $E \times A_2 \times A_2$, where $E \simeq A_2$ corresponds to an edge in the diagram containing the vertex Θ. $\psi(E)$ is a lattice point on the ray $\psi(\Gamma)$ by Th. 5.10, and so $u(F_1, F_2)$ must vanish on E. The only possibility is that the root on the edge E in Fig. 3 is in F_2. In our case we find that $456 \in F_1$ and $16, 34, 25 \in F_2$. Inspecting the list of D_4's in Remark 5.9, we see that the only possibility is $D(16, 34, 25)$ and

$$F_1 = \{145, 123, 246, 356\}, \quad F_2 = \{7, 16, 34, 25\}.$$

It is clear that $u(F_1, F_2)$ vanishes on the two remaining $A_2^{\times 3}$ because they contain only vertices of the pentadiagram other than 145 and edges other than $16, 34, 25$.

$\Theta = A_2^{\times 3}$. By symmetry Θ is given by edges 25, 46, and 13 of Figure 3. The D_4 and fourtuples are defined as follows. No vertex of the pentadiagram is in F_1 or F_2. But since $u(F_1, F_2)$ has valuation 3 on Θ, all edges of Θ have to be in F_1. It follows from Remark 5.9 that $D_4 = D(13, 25, 46)$ with fourtuples

$$F_1 = \{7, 13, 25, 46\}, \quad F_2 = \{124, 156, 236, 345\}.$$

Since neither F_1 nor F_2 contain other edges, $u(F_1, F_2)$ vanishes on the 4 remaining $A_2^{\times 3}$.

Proof for $\mathcal{G}'(E_7)$. In Figure 5 we plot all roots of all rays of $\mathcal{G}'(E_7)$. There are three S_4-orbits with $\Theta = A_1$: vertices, midpoints of edges, and the special root 247 of Lemma 8.4. There is one orbit with $\Theta = A_2$ (half-edges). There are two orbits with $\Theta = A_3^{\times 2}$. The first type corresponds to a pair of opposite edges, e.g. $135, 235, 237$ and $124, 126, 457$. We attach to each midpoint the only "extra" root in A_3 not contained in its A_1's or A_2's, e.g. we attach 137 to 235. The second type corresponds to a pair of vertices, e.g. $235, 135, 157$ and $367, 124, 126$ correspond to a pair $(135, 124)$. We attach extra roots in A_3's to midpoints (and align them in the direction of an edge connecting midpoints). Finally there is one orbit with $\Theta = A_7$: it corresponds to a pair of opposite edges (remove them from the tetrahedron to get the affine Dynkin diagram of A_7). Each A_7 contains 4 extra roots not contained in A_1, A_2, or A_3's and we plot them near the corresponding pair of opposite edges.

In each case we list the D_4 and the 4-tuples and show that they have an intrinsic characterization, i.e., they are preserved by the stabilizer $(S_4)_\Theta$. Thus it will suffice to prove (8.7.1) and to show that $|F_1 \cap \Gamma| = |F_2 \cap \Gamma|$ for a representative Γ of each $(S_4)_\Theta$-orbit. We leave this count (similar to the case of E_6) to the reader.

$\Theta = 247$ is an A_1 of Lemma 8.4. We take the vertices of the tetrahedron and the unique $D_4 = D(17, 25, 34)$ that contains them with fourtuples

$$\{247, 123, 357, 145\}, \quad \{6, 17, 25, 34\}.$$

$\Theta = 237$ is a vertex. We take $D_4 = D(34, 26, 57)$ (its Dynkin diagram is formed by Θ and midpoints of adjacent edges) with fourtuples

$$\{237, 245, 467, 356\}, \quad \{1, 34, 26, 57\}$$

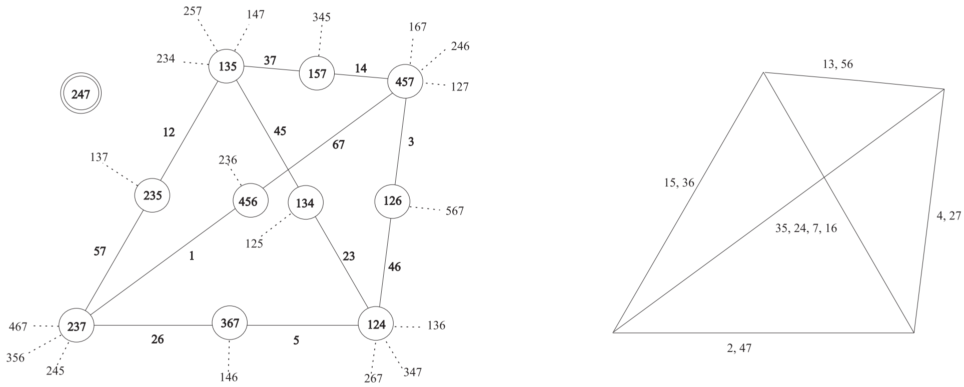


FIGURE 5.

$\Theta = 367$ is a midpoint. Take the two A_1 endpoints Γ_1 and Γ_2 of the edge (in our case 237 and 124). Take the unique $A_7 \not\ni \Theta$ and consider its four extra roots. Exactly two of them (2 and 56 in our case) are perpendicular to Γ_1 and Γ_2. This gives 4 roots that belong to a unique $D_4 = D(14, 37)$. We take the fourtuples

$$\{6, 25, 146, 367\}, \quad \{5, 26, 145, 357\}.$$

$\Theta = A_2$, e.g. given by a half-edge 26. A_2 is contained in three A_3's and their three extra (i.e. not contained in A_2's) roots (146, 245, and 356 in our case) along with the root on the half-edge 26 form a Dynkin diagram of $D_4 = D(26, 34, 15)$ with fourtuples

$$\{7, 15, 26, 34\}, \quad \{123, 146, 245, 356\}.$$

$\Theta = A_3^{\times 2}$ given by two vertices, e.g. 237 and 457. We consider the following 6 roots: extra roots of A_3's in (356 and 246), extra roots of $A_7 \supset \Theta$ that are not perpendicular to the two vertices (7 and 16), the midpoint of an edge connecting the two vertices (456), and the extra root of an A_3 formed by the two vertices and the midpoint above (236). Then there is only one D_4 containing all 6 roots, namely $D(16, 25, 34)$, and we take the fourtuples

$$\{7, 16, 34, 25\}, \quad \{123, 145, 356, 246\}.$$

$\Theta = A_3^{\times 2}$ given by a pair of disjoint edges, e.g., $135 - 157 - 457$ and $237 - 367 - 124$. There are two A_7's that contain Θ, and each of them has two extra roots perpendicular to midpoints of the disjoint edges (7, 24, 15, 36 in our case). There is only one D_4 that contains them, namely $D(24, 15, 36)$, and we take the fourtuples

$$\{123, 345, 146, 256\}, \quad \{7, 24, 15, 36\}.$$

$\Theta = A_7$, e.g., not containing 134 and 456. Consider 4 extra roots of A_7 (namely, 7, 16, 24, 35) and the unique $D_4 = D(16, 24, 35)$ that contains them, with fourtuples

$$\{7, 16, 24, 35\}, \quad \{123, 145, 346, 256\}.$$

□

§9. $\overline{Y}(\Delta)$ IS SMOOTH, TROPICAL, AND THE LOG CANONICAL MODEL

9.1. THEOREM. *Let $(\overline{Y}, B)$ be a pair of a (possibly reducible) proper variety with reduced boundary, which has stable toric singularities. Then $K_{\overline{Y}} + B$ is ample iff each irreducible open stratum is log minimal (in the reducible case strata are defined using boundary and double locus divisors).*

Proof. Since for an irreducible component $\overline{S} \subset \overline{Y}$ we have $(K_{\overline{Y}} + B)|_{\overline{S}} = K_{\overline{S}} + B_{\overline{S}}$, we may assume $\overline{Y}$ is irreducible. By adjunction, for any closed stratum $\overline{S}$ with interior S, $(K_{\overline{Y}} + B)|_{\overline{S}} = K_{\overline{S}} + B_{\overline{S}}$, where $B_{\overline{S}} = \overline{S} \setminus S$. Thus if $K_{\overline{Y}} + B$ is ample, S is log minimal and $S \subset \overline{S}$ is its log canonical model.

In the other direction we induct on $\dim \overline{Y}$. To show that $K_{\overline{Y}} + B$ is ample, we apply the Nakai–Moishezon criterion, i.e., we show that the restriction to every subvariety is big. By induction (and adjunction) it suffices to consider a subvariety which meets the interior $Y := \overline{Y} \setminus B$. But m-pluri log canonical forms of Y extend to any compactification $(\overline{Y}, B)$ with log canonical singularities, and so for some $m > 0$, the rational map

$$\overline{Y} \dashrightarrow \mathbb{P}(H^0(\overline{Y}, \mathcal{O}(m(K_{\overline{Y}} + B)))^*)$$

restricts to an immersion on Y, since by assumption Y is log minimal. Thus the restriction of the map to any subvariety meeting Y is birational. □

9.2. Corollary. *$\overline{Y}(E_6)$ and $\overline{Y}(D_n)$ are the log canonical models of their interiors.*

Proof. The compactifications have simple normal crossing boundary, so by Th. 9.1 we only need to check that open strata are all log minimal. Let $X(r, n)$ be the moduli space of ordered n-tuples of points in $\mathbb{P}^{r-1}$ in linear general position. All strata are open subsets of products of various $X(r, n)$ (indeed of $X(3, 6)$ or $X(2, i)$). This is familiar for $\overline{M}_{0,n}$. For $\overline{Y}(E_6)$ see [N]. An open subvariety of a log minimal variety is log minimal, and $X(r, n)$ is log minimal by [KT, 2.18]. □

Recall that, for $\operatorname{char} k \neq 2$, $\overline{Y}(E_7)$ denotes the compactification of $Y(E_7)$ given by $\tilde{M}_3^{(2)}$, the coarse moduli space of the blowup of the moduli stack of stable curves of genus 3 with level 2 structure at the T locus.

9.3. Fano Simplex. Let $\mathbb{P}^2(\mathbb{F}_2) = \{1 \dots 7\}$. Each of the 7 lines $\mathbb{P}^1(\mathbb{F}_2) \subset \mathbb{P}^2(\mathbb{F}_2)$ is given by a subset $\{ijk\}$ and gives rise to the root $\alpha_{ijk} \in E_7$. These roots are orthogonal because two lines intersect at a point. Fig. 6 shows one possible choice:

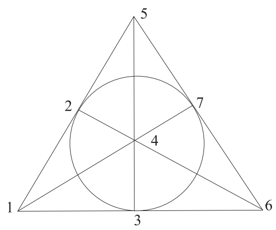


Figure 6. Fano plane

9.4. Lemma. *Let $A_1(\overline{Y}(E_7)) \subset \overline{Y}(E_7)$ be the complement to the union of boundary divisors not of type A_1. This open set, with its boundary, is smooth with simple normal crossings. It is covered by open subsets (V, B) of the following form. Consider the configuration of 6 lines in $\mathbb{P}^2$ as in Fig. 6 without the (impossible in* char $\neq 2$*) line $\{237\}$. Let $\tilde{V} \subset (\mathbb{P}^2)^7$ be the open subset of 7-tuples of points with no degenerations other than those in the configuration (so any triple of points other than the 6 triples in Fig. 6 are not collinear, and no 6 points lie on a conic).* PGL_3 *acts freely on $\tilde{V}$, let V be the quotient. Let $B \subset V$ be the union of the 6 boundary divisors.*

Proof. Over V there is a family of del Pezzo surfaces with at worse ordinary A_1 singularities (the anti-canonical model of the blowup of $\mathbb{P}^2$ at the given 7 points). The anti-canonical map for each fibre is a double cover of $\mathbb{P}^2$ branched over a quartic with at worst ordinary nodes. This determines a regular map $V \to \overline{M}_3 \setminus (\overline{H} \cup \delta_1)$.

9.5. Claim. *The map is surjective and quasi-finite.*

Proof. Choose $[C] \in \overline{\mathcal{M}}_3 \setminus (\overline{\mathcal{H}} \cup \delta_1)$. C embeds in $\mathbb{P}^2$ as a quartic nodal curve. Let $S' \to \mathbb{P}^2$ be the double cover branched over C, and $\tilde{S} \to S'$ the minimal desingularisation. The nodes of C give rise to a collection of disjoint (-2)-curves on $\tilde{S}$. By (4.9), it corresponds to a subsystem $A_1^{\times r} \subset E_7$, $r \leq 6$. We can extend this to an $A_1^{\times 6}$. Any two such are conjugate under $W(E_7)$ by Th. 7.4, which implies we can find a birational morphism $\tilde{S} \to \mathbb{P}^2$ realizing $\tilde{S}$ as the blowup at 7 distinct points, with only collinear degeneracies, as in the statement of the theorem. □

Let $Z \subset \tilde{M}_3^{(2)} \times V$ be the closure of the graph of the rational birational map $V \to \tilde{M}_3^{(2)}$. Then $Z \to V$ is proper and birational, and quasi-finite. Thus as V is normal, it is an isomorphism. So $V \to \tilde{M}_3^{(2)}$ is a birational quasi-finite morphism, and so an open immersion by Zariski's main theorem. Now we take the orbit of $V \subset \tilde{M}_3^{(2)}$ under $W(E_7)$. Each of these open sets admits an analogous description, and by the Claim they cover $p^{-1}(\overline{M}_3 \setminus \overline{H} \cup \delta_1)$, which is precisely $A_1(\overline{Y}(E_7))$. □

9.6. Lemma. *Let $(\mathcal{M}, \mathcal{D}_1 + \mathcal{D}_2 \cdots + \mathcal{D}_r)$ be a smooth Deligne-Mumford stack with a normal crossing divisor. Assume the restriction of the inertia stack to each of the substacks $\mathcal{D}_i$ contains a finite étale sub group-scheme, and that these sub-group schemes together generate the automorphisms of $\mathcal{M}$. Then the coarse moduli space $(M, D_1 + \ldots D_r)$ is smooth with normal crossings.*

Proof. The question is étale local, and it follows from the construction of the coarse moduli space in [KM] that for any geometric point $\operatorname{Spec} k \to \mathcal{M}$, with image $x \in M$, there exists an etale neighborhood U of x in M such that $\mathcal{M} \times_M U$ is isomorphic to a stack of the form $[X/G]$ for some scheme X, where G is the geometric stabilizer group of x. Using our assumptions, we can assume further that X is a vector space with a linear action of G, with a collection of G-invariant hyperplanes $\mathcal{E}_i$ ($[\mathcal{E}_i/G] = \mathcal{D}_i$), given by linearly independent linear functionals (X is the Zariski tangent space to $\mathcal{M}$ at a point). The conditions imply that G is generated by subgroups G_i acting trivially on $\mathcal{E}_i$. Clearly G_i is cyclic, generated by a reflection in $\mathcal{E}_i$. We can mod out by the intersection $\cap \mathcal{E}_i$ (on which G acts trivially) to reduce to the case when the linear functionals give a basis. Now we can choose coordinates so that G_i is generated by a diagonal matrix with all ones, except for a root of unity in the ith position. One checks the claim easily in this case. □

9.7. Lemma. *$\overline{Y}(E_7)$ has normal crossings generically along any codimension two boundary stratum contained in a divisor of type $D(A_7)$.*

Proof. We show the corresponding statement for $(\tilde{M}_3, B)$ holds. The result then follows from the proof Th. 4.8 – indeed the loops used in it are (homotopic to loops that are) contained in the automorphism free locus of M_3, and the proof of Th. 4.8 depends only on the monodromy action of these loops. Now we check the criteria of Lemma 9.6. The possible strata correspond to boundary strata of $\overline{H} = \overline{M}_{0,8}/S_8$, thus B_2, B_3 and B_4 as in [CH].

The generic point of B_2 corresponds to a nodal curve such that its normalization is a smooth genus 2 curve and a pair of points over the node is a conjugate pair for the (unique) hyperelliptic involution. It is clear that this nodal curve has automorphism group $\mathbb{Z}/2\mathbb{Z}$, the (restriction of the) hyperelliptic involution.

The generic point of B_3 corresponds to a 1-pointed elliptic curve glued to a 1-pointed hyperelliptic curve (of genus 2) at a Weierstrass point. In this case the automorphism group is $(\mathbb{Z}/2\mathbb{Z})^2$, generated by the restrictions of the involutions at the generic points of $\overline{H}$ and Δ_1.

The generic point of $B_4 = T \subset \overline{M}_3$ is the union of a pair of two 2-pointed genus 1 curves. The automorphism group is (generically) $\mathbb{Z}/2\mathbb{Z}$ – the involution on each component interchanging the marked points, the restriction of the hyperelliptic involution. The same then holds at $T = E \cap \tilde{H} \subset \tilde{M}_3$. $\square$

9.8. Lemma. *The restriction map* $r : M^{D(A_7)}(E_7) \to \mathcal{O}^*(M_{0,8})/k^* = M(D_8)$ *is surjective, where* $M^{D(A_7)}$ *means units with valuation* 0 *along* $D(A_7)$ *as in* (2.9)*.*

Proof. First we show that we can find a unit $u \in M^{D(A_7)}(E_7)$ which has valuation 1 along some boundary divisor meeting $D(A_7)$, i.e., the image of r is indivisible. Choose perpendicular subsystems A_1, A_1' of the given $A_7 \subset E_7$. It is easy to see using the explicit chart of Lemma 9.4 that we can find a unit with valuation 1 on A_1, and 0 on A_1'. If it has zero valuation on $D(A_7)$ we are done. Otherwise we can choose an element of the Weyl group which preserves the A_7 and interchanges A_1 and A_1'. Now take the difference between the original relation, and the one obtained from it by applying the permutation.

The map r is S_8-equivariant and by Lemma 5.3 the image is irreducible. $M(E_7)_{\mathbb{Q}}$ contains a single copy of $M(D_8)_{\mathbb{Q}}$ (e.g., by dimension count), and therefore it remains to show that there exists a surjective map $r' : M^{D(A_7)}(E_7) \to M(D_8)$ of $\mathbb{Z}[S_8]$-modules: surjectivity of r then follows from Schur's lemma. We choose the standard $A_7 \subset E_7$ (4.14.1). Let $f : \mathbb{Z}^{\mathcal{A}_1(E_7)} \to \mathbb{Z}^{\mathcal{A}_1(A_7)} \to \mathbb{Z}^{\mathcal{A}_1(D_8)}$ be the composite map, where the first map is the natural restriction and the second map sends $[\varepsilon_i - \varepsilon_j]$ to $[\varepsilon_i - \varepsilon_j] + [\varepsilon_i + \varepsilon_j]$. Here $i, j \in \{1, \dots, 8\}$ and we identify the root $\beta_{i8} \in A_7 \subset E_7$ with the root $\varepsilon_i - \varepsilon_8$ of the standard $A_7 \subset D_8$ of (7.2). Let r' be the restriction of f to $M^{D(A_7)}(E_7) \subset \mathbb{Z}^{\mathcal{A}_1(E_7)}$. We claim that r' is surjective onto $M(D_8) \subset \mathbb{Z}^{\mathcal{A}_1(D_8)}$. By Th. 5.7, any $m \in M^{D(A_7)}(E_7)$ is a linear combination of D_4-units, i.e., a linear combination of differences $F_1 - F_2$, where F_1 and F_2 are fourtuples of a D_4 of one of the three types of Remark 5.9. For D_4's of the first and third type, $F_1 - F_2 \in M^{D(A_7)}(E_7)$ (i.e., the coefficients at the roots in A_7 add up to 0) and $r'(F_1 - F_2)$ is a D_4-unit of D_8. For a D_4 of the second type, $F_1 - F_2$ (restricted to A_7) is either equal to 0 or is a linear combination of 4 orthogonal roots of A_7. It is easy to see that if $m \in M^{D(A_7)}(E_7)$ is a linear combination of such units then its restriction to A_7 can be rewritten using restriction to A_7 of D_4-units of the first and second type. Since $M(D_8)$ is also generated by D_4 units, we are done. $\square$

9.9. Lemma. $D(A_3 \times A_3) \subset \overline{Y}(E_7)$ *is isomorphic to* $\overline{M}_{0,5} \times \overline{M}_{0,5} \times \overline{M}_{0,4}$*, and the pullback of one boundary point from the third factor is* $D(A_1) \cap D(A_3 \times A_3)$ *where* $A_1 = (A_3 \times A_3)^{\perp}$*, while the pullback of the other two boundary points are* $D(A_7) \cap D(A_3 \times A_3)$ *for the two* A_7*'s which contain* $A_3 \times A_3$*.*

Proof. The divisor $D(A_3 \times A_3)$ is a component of $E^{(2)} \subset \tilde{M}_3^{(2)}$, so this follows from Lemma 4.16. $\square$

9.10. Theorem. *There is a commutative diagram*

$$\begin{array}{ccc} M(\Delta) & \xrightarrow{\oplus b_\Gamma} & \mathbb{Z}^{\mathcal{R}(\Delta)} \\ i\downarrow & & j\downarrow \\ \mathcal{O}^*(Y(\Delta))/k^* & \xrightarrow{\text{val}} & \mathbb{Z}^{\partial \overline{Y}(\Delta)} \end{array}$$

where val *sends a unit to the sum of its valuations at each of the boundary divisors, i is the identification of Th. 6.6, j is the identification of Lemma 7.2 and Prop. 7.4, and, for each type Γ of root system in $\mathcal{R}(\Delta)$, $b_\Gamma := c_\Gamma/m_\Gamma$, where c_Γ is the composition*

$$c_\Gamma : M(\Delta) \xrightarrow{\psi^\vee} \mathbb{Z}^{\Delta_+} \xrightarrow{A} \mathbb{Z}^{\Gamma(\Delta)}, \quad A([\alpha]) = \sum_{\alpha\in\Theta\in\Gamma(\Delta)} [\Theta]$$

and m_Γ is the largest integer such that $c_\Gamma(M(\Delta)) \subset m_\Gamma \mathbb{Z}^{\Gamma(\Delta)}$.

We define a collection of cones $\mathcal{F}(\partial\overline{Y}(\Delta)) \subset N(\Delta)_\mathbb{Q}$ from the combinatorics of the boundary of $\overline{Y}(\Delta)$ as in Cor. 2.4. The map $i^\vee$ identifies this collection with $\mathcal{F}(\Delta)$. In particular, $\mathcal{F}(\Delta)$ is supported on the tropicalization $\mathcal{A}(Y(\Delta))$.

Proof. The map dual to c_Γ has the form $c_\Gamma^\vee : [\Theta] \mapsto \psi(\Theta)$. It is clear from Th. 8.7 that $c_\Gamma \neq 0$ (and is therefore injective) and that m_Γ is given by Th. 5.10.

Let $\Gamma \in \mathcal{R}$ be a root subsystem, let $D(\Gamma) \in \partial\overline{Y}(\Delta)$ be the corresponding boundary divisor, and let $[D(\Gamma)] \in N_{Y(\Delta)}$ be the corresponding point. Let $\xi(\Gamma)$ be the first lattice point of $N(\Delta)$ along the ray spanned by $\psi(\Gamma)$. Note commutativity of the diagram is equivalent to $i^\vee([D(\Gamma)]) = \xi(\Gamma)$. Assuming this, the equality of the collections of cones $\mathcal{F}(\Delta)$ and $\mathcal{F}(\partial\overline{Y}(\Delta))$ follows from Lemma 7.2 and Prop. 7.4. Let $H \subset W(\Delta)$ be the normalizer of Γ (and thus of $D(\Gamma)$). Note that $\psi(\Gamma)$ and $[D(\Gamma)]$ are H-invariant.

9.11. Claim. $\dim N(\Delta)^H = 1$.

Proof. Let $\Delta = D_n$. We use the equivariant surjection $\psi : \mathbb{Z}^{\mathcal{A}_1} \to N$ of Lemma 5.2. $H = W(D_k) \times W(D_{n-k})$ has three orbits on $\mathcal{A}_1$, namely $\mathcal{A}_1(D_k)$, $\mathcal{A}_1(D_{n-k})$, and their complement. It follows that $N(\Delta)^H$ is generated by $\psi(D_k)$, $\psi(D_{n-k})$, and $\psi(D_n) - \psi(D_k) - \psi(D_{n-k})$. The first two vectors are equal (see the proof of Th. 5.10), and $\psi(D_n) = 0$ (being a W-invariant vector in an irreducible module).

If $\Delta = E_6$ or E_7 then we can either make a similar calculation or note that, by Frobenius reciprocity, $\dim N(\Delta)^H$ is equal to the multiplicity of $N(\Delta)$ in the permutation module $\mathbb{Z}^{\Gamma(\Delta)}$. The latter is equal to 1 by the tables of [A]. □

It follows that vectors $\psi(\Gamma)$ and $[D(\Gamma)]$ are collinear.

9.12. Claim. *The vectors $\psi(\Gamma)$ and $[D(\Gamma)]$ are not opposite.*

Proof. We know that $[D(\Gamma)] \in \mathcal{A}_{Y(\Delta)}$ by (2.4). We claim that $\psi(\Gamma)$ also belongs to the tropicalization. By Th. 6.6, the dominant map $\mathcal{X}(\Delta) \xrightarrow{\Psi} Y(\Delta)$ induces the linear map $\mathbb{Z}^{\Delta_+} \xrightarrow{\psi} N(\Delta)$ that restricts to the map of tropicalizations $\mathcal{A}(\mathcal{X}) \to \mathcal{A}(Y)$. The fan $\tilde{\mathcal{F}}(\Delta)$ supported on $\mathcal{A}(\mathcal{X})$ is described in [ARW] but we only need the description of rays: they correspond to irreducible root subsystems $\tilde{\Theta} \subset \Delta$ such that $\Gamma(\tilde{\Theta}) \subset \Gamma(\Delta)$. The corresponding ray of $\tilde{\mathcal{F}}(\Delta)$ is given by $\sum_{\alpha\in\tilde{\Theta}^+}[\alpha]$. Therefore, it suffices to find, for each $\Theta \in \mathcal{R}(\Delta)$, a subsystem $\tilde{\Theta} \subset \Delta$ realizable on the Dynkin diagram and such that $\psi(\Theta)$ and $\psi(\tilde{\Theta})$ are proportional. This is easy using the table of Theorem 5.10.

Now we argue by contradiction. If $\psi(\Gamma)$ and $[D(\Gamma)]$ are opposite then $\mathcal{A}(Y(\Delta))$ contains a line L. Take any $D_4 \subset \Delta$. Then $\mathcal{A}(Y(\Delta))$ maps onto $\mathcal{A}(Y(D_4))$ by Theorem 6.7. Since $\mathcal{A}(Y(D_4)) = \mathcal{A}(M_{0,4})$ does not contain lines (being the union of three rays), L maps to 0 in $N(D_4)$. But by Theorem 5.7, the map $N(\Delta) \to \bigoplus_{D_4\subset\Delta} N(D_4)$ is injective. This is a contradiction. □

To complete the proof it's enough to show that $[D(\Gamma)]$ is the first lattice point along its ray. For this its enough to find some unit with valuation one along this divisor. For $\overline{Y}(D_n) = \overline{M}_{0,n}$ this is easily checked — one can take a classical cross-ratio of some 4-tuple, see [K]. For $\overline{Y}(E_6) = \overline{Y}^6$ (where there are only two orbits of divisors) we can take any of Naruki's cross-ratio maps, see [N].

Now assume $\Delta = E_7$. Suppose first that $\Gamma \neq A_7$. We choose an A_7 such that $D(A_7)$ intersects $D(\Gamma)$. Note if we have a unit u with valuation zero along $Y(A_7)$, then, by Lemma 9.7, its valuation along $D(\Gamma)$ is the same as the valuation of $u|_{D(A_7)}$ along $D(\Gamma) \cap D(A_7)$. As above, on $D(A_7) = \overline{M}_{0,8}$ we can find a unit with valuation one along any given boundary divisor. So the result follows from Lemma 9.8.

To finish the proof it suffices to find a unit with valuation one along some $Y(A_7)$. Suppose it does not exist. Then, by equivariance, valuation of any unit along any $D(A_7)$ is divisible by $d > 1$. Consider subsystems $A_3 \times A_3$ and $A_1 := (A_3 \times A_3)^{\perp}$. By Th. 9.14 (and the previous case $\Gamma \neq A_7$), we can find a unit u that has valuation 0 along $D(A_3 \times A_3)$ and valuation 1 along $D(A_1)$. By Lemma 9.9, the valuations of u at the two A_7 boundary divisors that meet $D(A_3 \times A_3)$ add up to -1, and therefore must be coprime. This contradicts our assumption. □

9.13. Corollary. *$\overline{Y}(E_7) = (\tilde{M}^{(2)}, B)$ is smooth with simple normal crossings.*

Proof. By the strict simpliciality of the cones $\sigma \in \mathcal{F}(\Delta)$ (Th. 9.14), each boundary divisor is Cartier: this is a local question, and by the strict simpliciality, given a collection of boundary divisors cutting out a stratum S, we can find a unit with valuation one on one of them, and zero on the others — and so a boundary divisor is locally principal. Our calculations have shown each boundary divisor is smooth. Thus $\overline{Y}(E_7)$ is smooth. Since each boundary divisor has simple normal crossing boundary induced (by the strict simpliciality) by the boundary of $\overline{Y}(E_7)$, we have simple normal crossings for $\overline{Y}(E_7)$. □

9.14. Theorem. *Let $\mathcal{P} = \prod_{D_4 \subset \Delta} \mathcal{F}(D_4) \subset N' = \bigoplus_{D_4 \subset \Delta} N(D_4)$ be the product fan. $\mathcal{F}(\Delta)$ is a convexly disjoint and strictly simplicial fan, and each of its cones is a cone in the intersection fan $N(\Delta)_{\mathbb{Q}} \cap \mathcal{P}$, where we identify $N(\Delta)$ with a sublattice in N' using Th. 5.7. The map of toric varieties $X(\mathcal{F}) \to X(\mathcal{P})$ is an immersion, i.e., an isomorphism of $X(\mathcal{F})$ with an open subscheme of a closed subscheme of $X(\mathcal{P})$.*

Proof. Let σ be a cone of $\mathcal{F}(\Delta)$. By Cor. 6.7, each map $N(\Delta) \to N(D_4)$ is induced by the morphism of very affine varieties $Y(\Delta) \to Y(D_4)$. So by Th. 2.5 we have a surjective map of tropical sets $|\mathcal{F}(\Delta)| \to |\mathcal{F}(D_4)|$. Since $\mathcal{F}(D_4)$ is one-dimensional, the image of any face of σ is a cone of $\mathcal{F}(D_4)$. By Lemma 8.6, σ is a cone of $\mathcal{G}(\Delta)$. It follows (by Th. 8.7) that $\operatorname{Ann}\langle\sigma\rangle \subset M(\Delta)_{\mathbb{Q}}$ is generated by the restriction of elements from the various $M(D_4)$, where $\langle\sigma\rangle \subset N(\Delta)_{\mathbb{Q}}$ means the linear span and $\operatorname{Ann}\langle\sigma\rangle \subset M(\Delta)_{\mathbb{Q}}$ is the annihilator $\operatorname{Ker}[M(\Delta)_{\mathbb{Q}} \to \langle\sigma\rangle^{\vee}]$. Now it follows from Lemma 9.16 below that σ is a cone of the intersection fan.

It follows that $\mathcal{F}(\Delta)$ is a fan. It is strictly simplicial by Th. 8.7 and Lemma 8.6. To show that it is convexly disjoint we note that: (1) $\mathcal{F}(D_4)$ is convexly disjoint; (2) the product of convexly disjoint collections is convexly disjoint; (3) the intersection of a convexly disjoint collection with a linear subspace is convexly disjoint.

Let U be the T_N-toric variety given by the intersection fan $N(\Delta)_{\mathbb{Q}} \cap \mathcal{P}$. It contains $X(\mathcal{F})$ as a toric open subset. Since N is saturated in N' by Th. 5.7, U is the normalisation of its image under the the natural map $\nu : U \to X(\mathcal{P})$, and ν induces a bijection of the sets of T_N orbits of U and $\nu(U)$ [O, 1.4]. Therefore it suffices to prove that $\nu(U(\sigma))$ is normal for any affine open toric subset $U(\sigma)$ given by a cone $\sigma \in \mathcal{F}(\Delta)$. Let $\sigma' = \prod_i \pi_i(\sigma) \subset N'_{\mathbb{Q}}$, where $\pi_i : N \to N(D_4)$ are the projections. We claim that $U(\sigma) \to U(\sigma')$ is a closed embedding. It suffices to prove that the map of semigroups $(\sigma')^{\vee} \cap M' \to \sigma^{\vee} \cap M$ is surjective. But this follows from Th. 8.7. □

9.15. Remark. Th. 9.14 holds if we replace $\mathcal{F}$ by $\mathcal{G}$ everywhere (in particular, $\mathcal{G}$ is itself a fan). We don't need this here. The same proof holds if one can establish that under $N(\Delta) \to N(D_4)$ each cone of $\mathcal{G}(\Delta)$ maps onto a cone of $\mathcal{G}(D_4)$. Since this is true of the one dimensional cones (which are part of $\mathcal{F}$) it's enough to show this is a map of fans. This follows from Sekiguchi's descriptions of the real cells (which appear without proof in the E_7 case) – as it's clear (by continuity) that $Y(\Delta)(\mathbb{R}) \to Y(D_4)(\mathbb{R}) = M_{0,4}(\mathbb{R})$ sends a cell into a cell.

9.16. Lemma. *Let $N \subset \bigoplus_i N_i$ be a sub-lattice of a finite direct sum of lattices. Let $\sigma \subset N_{\mathbb{Q}}$ be a rational polyhedral cone. Assume that for each face $\gamma \subset \sigma$ and each projection $\pi_i : N \to N_i$, $\pi_i(\gamma)$ is a face of $\pi_i(\sigma)$. Then $\sigma = N_{\mathbb{Q}} \cap \prod_i \pi_i(\sigma)$ if for each face $\gamma \subset \sigma$ the natural map $\bigoplus_i \operatorname{Ann}\langle \pi_i(\gamma)\rangle \to \operatorname{Ann}\langle\gamma\rangle$ is surjective.*

Proof. For any face $\gamma \subset \sigma$ let $P(\gamma) = \prod_i \pi_i(\gamma)$ and $I(\gamma) = P(\gamma) \cap N_{\mathbb{Q}}$. By assumption, $P(\gamma)$ is a face of $P(\sigma)$ and so $I(\gamma)$ is a face of $I(\sigma)$. $\gamma \subset I(\gamma)$ and, if we can show that they have the same dimension, then every face of σ lies in a face of $I(\sigma)$ of the same dimension and therefore $\sigma = I(\sigma)$. It suffices to check that $\langle\gamma\rangle = \langle I(\gamma)\rangle$, i.e., $\langle\gamma\rangle = N_{\mathbb{Q}} \cap \bigoplus_i \langle \pi_i(\gamma)\rangle$. But this clearly follows from the surjectivity of $\bigoplus_i \operatorname{Ann}\langle\pi_i(\gamma)\rangle \to \operatorname{Ann}\langle\gamma\rangle$. □

9.17. Proposition. *Let $\Gamma \in \mathcal{R}(\Delta)$. Th. 9.10 induces the identification of boundary divisors of $D(\Gamma)$ with $\operatorname{Link}_\Gamma(\mathcal{R}(\Delta))$. It has the following explicit form:*

$\underline{A_1 \subset E_6.}$ $\operatorname{Link}_\Gamma(\Delta)$ is given by A_1 and $A_2^{\times 2}$ subsystems of $\Gamma^\perp = A_5$. In the notation of Lemma 7.2, these correspond to $(2,4)$ and $(3,3)$ bipartitions of N_6 respectively, which correspond to boundary divisors of $D(\Gamma) = \overline{M}_{0,N_6}$.

$\underline{A_2^{\times 3} \subset E_6.}$ The link is given by A_1 subsystems of Γ. Identifying each factor with a single standard A_2, and fixing $N_3 \subset N_4$, each A_1 gives a two element subset of one of three copies of $N_3 \subset N_4$, and thus a boundary divisor of $D(\Gamma) = \overline{M}_{0,N_4}^{\times 3}$.

$\underline{A_7 \subset E_7.}$ The link consists of A_1, A_2, and $A_3^{\times 2} \subset A_7$. These correspond to $(2,6)$, $(3,5)$ or $(4,4)$ bipartitions of N_8, and thus boundary divisors of $D(\Gamma) = \overline{M}_{0,8}$.

$\underline{A_2 \subset E_7.}$ Identify Γ and $\Gamma^\perp = A_5$ with the standard A_2 and A_5. Fix $N_3 \subset N_4$ and $N_6 \subset N_7$. The link consists of $A_1 \subset \Gamma$ (boundary divisors of $\overline{M}_{0,4}$) and of $A_1 \subset \Gamma^\perp$, $A_2 \subset \Gamma^\perp$, $A_3^{\times 2} \supset \Gamma$, or $A_7 \supset \Gamma$. The last two possibilities are equivalent to $A_3 := A_3^{\times 2} \cap \Gamma^\perp \subset \Gamma^\perp$ and $A_4 := A_7 \cap \Gamma^\perp \subset \Gamma^\perp$, respectively. These subsystems correspond to k-element subsets of N_6, $2 \le k \le 5$, and thus to boundary divisors of the second factor of $D(\Gamma) = \overline{M}_{0,4} \times \overline{M}_{0,7}$.

$\underline{A_3^{\times 2} \subset E_7.}$ The link consist of A_1 or A_2 subsystems of the two factors, $A_7 \supset \Gamma$ or $A_1 = \Gamma^\perp$. The last two possibilities correspond to the boundary divisors from the third factor of $D(\Gamma) = \overline{M}_{0,5} \times \overline{M}_{0,5} \times \overline{M}_{0,4}$ and the other possibilities correspond (as above) to boundary divisors of the first two factors.

Proof. It is easy to check using Table 1 that the given map is a $W(\Delta)_\Gamma$-equivariant bijection. Now we claim that $\operatorname{Aut}_{W_\Gamma}(\operatorname{Link}_\Gamma \mathcal{R})$ is trivial except for $A_3^{\times 2} \subset E_7$, in which case it is equal to $\mathbb{Z}/2\mathbb{Z}$ induced by the involution of $D(\Gamma)$ given by the reflection in $A_1 := (A_3^{\times 2})^\perp$. Indeed, let $G := W(\Gamma \times \Gamma^\perp)$ Then $G \subset W_\Gamma$ and therefore if $f \in \operatorname{Aut}_{W_\Gamma}(\operatorname{Link}_\Gamma \mathcal{R})$ then $G_\Theta = G_{f\Theta}$ for any Θ in the link. Since $\Gamma \times \Gamma^\perp$ has at most one direct summand A_1, a subsystem $\Theta \subset \Gamma \times \Gamma^\perp$ of type A_1 is uniquely determined by G_Θ. Since any $A_1 \subset \Gamma \times \Gamma^\perp$ belongs to the link, f must preserve all of them. It follows that f preserves all subsystems in the link that belong to $\Gamma \times \Gamma^\perp$ because they are uniquely determined by incident A_1's in the link. A simple calculation shows that in fact f preserves all subsystems in the link, except when $\Gamma = A_3^{\times 2}$: in this case it is possible to permute the two A_7's that contain Γ. This permutation is induced by the reflection in $A_1 := (A_3^{\times 2})^\perp$. □

9.18. Theorem. *Let $\Delta = D_n$ or E_n, $n \le 7$. $Y(\Delta)$ is Hübsch, $\overline{Y}(\Delta)$ is the log canonical model, and $\mathcal{F}(\Delta)$ is the log canonical fan. The rational map*

$$\overline{Y}(\Delta) \to P := \prod_{D_4 \subset \Delta} \overline{Y}(D_4) \tag{9.18.1}$$

is a closed embedding. Each open (resp. closed) boundary stratum is the scheme-theoretic inverse image of the open (resp. closed) stratum of P which contains it.

9.19. Remark. Th. 9.18 and Cor. 9.13 imply Sekiguchi's Conjecture 1.5.

Proof. All statements about the embedding of $\overline{Y}(E_7)$ into the product of $\mathbb{P}^1$'s follow from Th. 9.14 once we prove that $\mathcal{F}(\Delta)$ is the log canonical fan.

$Y(\Delta) \subset \overline{Y}(\Delta)$ is a simple normal crossing compactification by Cor. 9.13. The open strata are $Y(\Delta)$, $D^0(A_1) \subset \overline{Y}(E_7)$, or products of $M_{0,i}$'s. They are log minimal as open subsets of the complements to connected hyperplane arrangements (for $D^0(A_1)$ this follows from Lemma 9.4). It follows that $\overline{Y}(\Delta)$ is the log canonical model (Th. 9.1). We prove $Y(\Delta)$ is Hübsch by showing for each stratum the map $S \to T_Y^S$ of Th. 2.10 is an immersion. For $Y(\Delta)$ this is clear, as $Y(\Delta)$ is very affine. For purely A_1 strata of $\overline{Y}(E_7)$ this is easily checked using the explicit charts of Lemma 9.4. All other strata are products of $M_{0,i}$, which are very affine. So it's enough by Th. 2.10 to show that $M^S_{Y(\Delta)} \to M_S$ is surjective for such strata.

Proof for D_n: by Cor. 6.7 and Th. 5.7, M_S is generated by the pullback of units from the canonical cross-ratio maps $S \to M_{0,4}$. Thus it is enough to show that each such map is the restriction of a map $M_{0,n} \to M_{0,4}$. This is well known.

Proof for E_6 and E_7: all boundary divisors (other than $D(A_1) \subset \overline{Y}(E_7)$) are products of $\overline{M}_{0,i}$, and so, by the D_n case, to prove $M^S_{Y(\Delta)} \to M_S$ is surjective it's enough to consider the case where S is codimension one. Let $r := \operatorname{rk} M_S$ (given in Lemma 5.3). It's enough to find r boundary divisors $D_1, \dots, D_r$, all incident to $\overline{S}$, and units $u_1, \dots, u_r$ so that $\operatorname{val}_{D_i}(u_j) = \delta_{ij}$, and $\operatorname{val}_S(u_i) = 0$, for then $u_i \in M_Y^S$, and their images in M_S will give a basis of the lattice. We choose a maximal cone $\sigma \in \mathcal{G}(\Delta)$ which has a ray corresponding to the boundary divisor S. By Th. 8.7, it suffices to prove the following claim:

9.20. Claim. *The cone σ contains r rays corresponding to divisors (other than S) which have non-empty intersection with $\overline{S}$.*

9.21. Remark. This follows from Th. 8.7 (applied to the boundary divisor) if we know the rays in a cone indeed correspond to the faces of a cell, as the intersection of a cell with a boundary divisor will be a cell for the corresponding compactification.

We check the claim for each $W(\Delta)$ orbit of boundary divisor, using Fig. 3. Proof for E_6: an $A_2^{\times 3}$ subdiagram of the pentadiagram contains 6 A_1's which yield $u_1, \dots, u_6$ for the boundary divisor $\overline{M}_{0,4}^{\times 3} = D(A_2^{\times 3}) \subset \overline{Y}(E_6)$. An A_1 subdiagram is contained in 3 $A_2^{\times 3}$ subdiagrams and is perpendicular to 6 A_1's. This gives 9 units for the boundary divisor $\overline{M}_{0,6} = D(A_1) \subset \overline{Y}(E_6)$. Proof for E_7: an A_7 of the tetra-diagram contains 8 A_1's, 8 A_2's, and 4 $A_3^{\times 2}$'s, which yield $u_1, \dots, u_{20}$ for the boundary divisor $\overline{M}_{0,8} = D(A_7) \subset \overline{Y}(E_7)$. Consider an $A_3^{\times 2}$ subdiagram that corresponds to the pair of opposite edges. It contains 6 A_1's and 4 A_2's, and is contained in 2 A_7's. Another type of an $A_3^{\times 2}$ subdiagram contains 6 A_1's and 4 A_2's, is contained in an A_7, and is perpendicular to an A_1 from the tetradiagram. In both cases this gives 12 units of $M_{0,5} \times M_{0,5} \times M_{0,4}$. Finally, consider an A_2 subdiagram. It contains 2 A_1's, is perpendicular to 5 A_1's and 4 A_2's, and is contained in 3 $A_3^{\times 2}$'s and 2 A_7's. This gives 16 units of $M_{0,4} \times M_{0,7}$. □

§10. Moduli of stable Del Pezzo surfaces

In this section we explicitly construct the compactification of the moduli space Y^n of marked del Pezzo surfaces given by moduli of stable surfaces with boundary for $n \leq 6$.

10.1. Proposition. *Let $\Delta' \subset \Delta$ be either $D_n \subset D_{n+1}$, or $E_n \subset E_{n+1}$ for $n \leq 6$. The map $\pi : N(\Delta) \to N(\Delta')$ of Lemma 5.4 induces a map of fans $\mathcal{F}(\Delta) \to \mathcal{F}(\Delta')$. Let $p : Y(\Delta) \to Y(\Delta')$ be the following map: For D_n, $M_{0,n+1} \to M_{0,n}$ is the forgetful map. For E_n, $Y^{n+1} \to Y^n$ is (6.3.2). Then $\pi^\vee$ is the pullback of units $p^* : \mathcal{O}(Y(\Delta'))/k^* \to \mathcal{O}(Y(\Delta))/k^*$. We have a commutative diagram of morphisms*

$$\begin{array}{ccccc} \mathcal{X}(\Delta) & \xrightarrow{\Psi} & Y(\Delta) & \longrightarrow & X(\mathcal{F}(\Delta)) \\ {\scriptstyle \mathrm{pr}}\downarrow & & {\scriptstyle p}\downarrow & & {\scriptstyle \pi}\downarrow \\ \mathcal{X}(\Delta') & \xrightarrow{\Psi'} & Y(\Delta') & \longrightarrow & X(\mathcal{F}(\Delta')) \end{array}$$

where pr *is the projection dual to the embedding of root lattices $\Lambda(\Delta') \hookrightarrow \Lambda(\Delta)$.*

Proof. Under the identifications of Th. 6.6 (for Δ and Δ'), the pullback of units p^* is identified with $\pi^\vee$ by commutativity of (5.4.1). By Th. 2.5, p induces a surjective map of tropicalizations $\mathcal{A}_{Y(\Delta)} \to \mathcal{A}_{Y(\Delta')}$. This map is automatically a map of fans $\mathcal{F}(\Delta) \to \mathcal{F}(\Delta')$ by Th. 1.10. It induces the map of toric varieties and therefore (by Th. 9.18) the regular map $\overline{Y}(\Delta) \to \overline{Y}(\Delta')$. □

Next we describe $\pi : \mathcal{F}(\Delta) \to \mathcal{F}(\Delta')$ explicitly. For any $[\Gamma] \in \mathcal{R}$, we denote by $\zeta(\Gamma)$ the first lattice point along the corresponding ray of $\mathcal{F}$.

10.2. Proposition. *$\underline{D_n \subset D_{n+1}}$. Rays of $\mathcal{F}(\Delta)$ correspond to bipartitions of N_{n+1},*

$$\pi(\zeta(I \sqcup I^c)) = \begin{cases} \zeta((I \cap N_n) \sqcup (I^c \cap N_n)) & \text{if } |I \cap N_n|,\ |I^c \cap N_n| > 1 \\ 0 & \text{otherwise} \end{cases} \tag{10.2.1}$$

$\underline{E_5 := D_5 \subset E_6}$. For $A_1 \in \mathcal{R}(E_6)$,

$$\pi(\zeta(A_1)) = \begin{cases} \zeta(D_2 \subset D_5) & \text{if } A_1 \subset D_5 \\ 0 & \text{otherwise (there are 16 of those)} \end{cases} \tag{10.2.2}$$

We use that, in D_5, $A_1 \times A_1^\perp = (A_1 \times A_1') \times D_3 = D_2 \times D_3$. For $A_2^{\times 3} \in \mathcal{R}(E_6)$,

$$\pi(\zeta(A_2^{\times 3})) = \zeta(D_2 \subset D_5), \text{ where } A_2^{\times 3} \cap D_5 = A_2 \times A_1^{\times 2} = A_2 \times D_2. \tag{10.2.3}$$

In both cases the toric map $X(\mathcal{F}(\Delta)) \to X(\mathcal{F}(\Delta'))$ is flat, with reduced fibres.

Proof. Rays of $\mathcal{F}(D_n)$ correspond to bipartitions by Lemma 7.2. The first lattice points are found in Th. 5.10 (see also the proof of Th. 8.7). The identities (10.2.1–10.2.3) follow from Lemma 5.4 and Th. 5.10. The identity $A_2^{\times 3} \cap D_5 = A_2 \times A_1^{\times 2}$ is easy to check. The last statement follows from Lemma 10.3. □

10.3. Lemma. *Let $N \to N'$ be a surjection of lattices giving a map of fans $\mathcal{F} \to \mathcal{F}'$. Assume $\mathcal{F}'$ is strictly simplicial and any ray of $\mathcal{F}$ maps to a ray of $\mathcal{F}'$. Then the toric map $X(\mathcal{F}) \xrightarrow{\pi} X(\mathcal{F}')$ is flat. If, in addition, the first lattice point of any ray of $\mathcal{F}$ maps to the first lattice point of a ray in $\mathcal{F}'$ (or to 0) then π has reduced fibres.*

Proof. Since $\mathcal{F}'$ is strictly simplicial, every cone in $\mathcal{F}$ maps onto a cone of $\mathcal{F}'$, which implies that π is equidimensional. Since $X(\mathcal{F}')$ is regular and $X(\mathcal{F})$ is Cohen–Macaulay [Da], this implies that π is flat [EGA4, 6.1.5]. The condition on lattice points implies that the scheme-theoretic inverse image of any toric stratum in $X(\mathcal{F}')$ is reduced. This implies that all fibers are reduced by equivariance. □

10.4. Definition. Let $\tilde{\mathcal{F}}(E_6)$ be the refinement of $\mathcal{F}(E_6)$ obtained by performing the barycentric subdivision of cones of type $\psi\{A_1,\dots,A_1\}$ and taking the corresponding minimal subdivision of cones of type $\psi\{A_2^{\times 3},A_1,\dots,A_1\}$ (meaning we barycentrically subdivide the codimension one face spanned by the A_1 rays, and then add the final $A_2^{\times 3}$ ray to each cone in this subdivision).

10.5. Proposition. *For $E_6\subset E_7$, π has the following effect on the rays of $\mathcal{F}(E_7)$:*

$$\pi(\zeta(A_1)) = \begin{cases}\zeta(A_1\subset E_6) & \text{if } A_1\subset E_6\\ 0 & \text{otherwise (there are 27 of those);}\end{cases} \tag{10.5.1}$$

$$\pi(\zeta(A_2)) = \begin{cases}\zeta(A_1\subset E_6) & \text{if } A_2\cap E_6 = A_1\\ \zeta(A_2^{\times 3}) & \text{if } A_2\subset E_6,\ \text{where } A_2^{\times 3} = A_2\times A_2^{\perp};\end{cases} \tag{10.5.2}$$

$$\pi(\zeta(A_3^{\times 2})) = \begin{cases}\zeta(A_2^{\times 3}) & \text{if } A_3^{\times 2}\cap E_6 = A_2^{\times 2},\ \text{here } A_2^{\times 3} := A_2^{\times 2}\times (A_2^{\times 2})^{\perp}\\ \zeta(A_1)+\zeta(A_1') & \text{if } A_3^{\times 2}\cap E_6 = A_3\times A_1\times A_1';\end{cases} \tag{10.5.3}$$

$$\pi(\zeta(A_7)) = \zeta(A_1\subset E_6),\ \text{where } A_7\cap E_6 = A_1\times A_5. \tag{10.5.4}$$

The map $X(\mathcal{F}(E_7))\to X(\mathcal{F}(E_6))$ has reduced fibres.

The fan $\tilde{\mathcal{F}}(E_6)$ is strictly simplicial. It is the unique minimal refinement of $\mathcal{F}(E_6)$ so that $\pi(\sigma)$ is a union of cones for each $\sigma\in\mathcal{F}(E_7)$. The collection of cones

$$\tilde{\mathcal{F}}(E_7) = \{\pi^{-1}(\gamma)\cap\sigma \mid \gamma\in\tilde{\mathcal{F}}(E_6),\ \sigma\in\mathcal{F}(E_7)\}$$

is a fan. The map $X(\tilde{\mathcal{F}}(E_7))\to X(\tilde{\mathcal{F}}(E_6))$ is flat, with reduced fibres.

Proof. The first lattice points are found in Th. 5.10 (see also the proof of Th. 8.7). Formulas for the image of π follow from Lemma 5.4 and Th. 5.10 up to (easily checked) formulas for the intersection of root subsystems of E_7 with E_6.

The only case when the image of a ray in $\mathcal{F}(E_7)$ is contained in the relative interior of a cone in $\mathcal{F}(E_6)$ is $\psi\{A_3^{\times 2}\}$ mapped onto the bisector of $\psi\{A_1,A_1'\}$. Fig. 7 shows a cone $\sigma\in\mathcal{F}(E_7)$ such that $\pi(\sigma)$ is a 4-dimensional cone with rays $\psi\{A_1'\},\psi\{A_1''\},\psi\{A_1'''\}$, and the bisector of $\psi\{A_1,A_1'\}$. It easily follows that $\tilde{\mathcal{F}}(E_6)$ is the minimal refinement such that $\pi(\sigma)$ is a union of cones for each $\sigma\in\mathcal{F}(E_7)$.

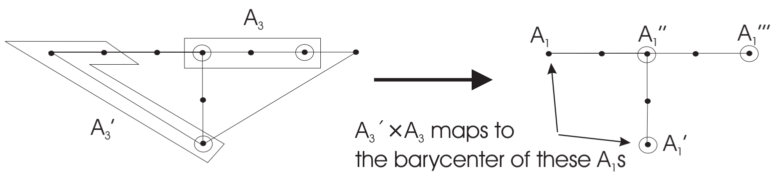


Figure 7.

It is clear $\tilde{\mathcal{F}}(E_7)$ is a fan, and that a cone of $\tilde{\mathcal{F}}(E_7)$ (as a subset of the lattice!) maps onto a cone of $\tilde{\mathcal{F}}(E_6)$. As the latter is strictly simplicial, the map of toric varieties is flat, with reduced fibres, by Lemma 10.3. □

10.6. Definition. A very affine variety Y is called *rigid in its intrinsic torus T_Y* if

$$\{t\in T_Y \mid t\cdot Y\subset Y\} = \{e\}.$$

10.7. Lemma. *The complement to a connected hyperplane arrangement is rigid in its intrinsic torus. Any open boundary stratum of $\overline{Y}(E_6)$ is rigid in its intrinsic torus.*

Proof. For the first statement, see [T, §4]. The boundary strata of $\overline{Y}(E_6)$ are all products of $M_{0,k}$ and thus complements to connected hyperplane arrangements. □

10.8. Lemma. *Let $\overline{Z} \xrightarrow{f} \overline{Y}$ be a surjective morphism of normal proper varieties with divisorial boundary such that $f^{-1}(\partial\overline{Y}) \subset \partial\overline{Z}$. The pullback of boundary divisors $\mathbb{Z}^{\partial\overline{Y}} \xrightarrow{f^*} \mathbb{Z}^{\partial\overline{Z}}$ determines f if $Y := \overline{Y} \setminus \partial\overline{Y}$ is very affine and rigid in its intrinsic torus.*

Proof. The map $\mathcal{O}^*(Y)/k^* \to \mathcal{O}^*(Z)/k^*$ is determined by the pullback of boundary divisors (by pulling back the divisor of zeroes and poles of $u \in \mathcal{O}^*(Y)$). This determines the composition $Z \to Y \to T_Y$ up to a translation in T_Y preserving Y. □

10.9. Proposition. *In the notation of Prop. 10.2 and Prop. 10.5, we describe the restriction $q := p|_{D(\Gamma)}$ of $\overline{Y}(\Delta) \xrightarrow{p} \overline{Y}(\Delta')$ to $D(\Gamma) \subset \overline{Y}(\Delta)$ for $\Gamma \in \mathcal{R}(\Delta)$. Below all maps $\overline{M}_{0,i} \to \overline{M}_{0,j}$, $j \leq i$ are canonical fibrations given by dropping points.*

$$\text{for (10.2.2):} \quad q = \begin{cases} \overline{M}_{0,6} \to \overline{M}_{0,4} = D(D_2 \subset D_5) \\ \overline{M}_{0,6} \to \overline{M}_{0,5} = \overline{Y}(D_5); \end{cases}$$

$$\text{for (10.2.3):} \quad q : \ \overline{M}_{0,4} \times \overline{M}_{0,4} \times \overline{M}_{0,4} \xrightarrow{\mathrm{pr}_1} \overline{M}_{0,4} = D(D_2 \subset D_5);$$

$$\text{for (10.5.2):} \quad q = \begin{cases} \overline{M}_{0,4} \times \overline{M}_{0,7} \to \{pt\} \times \overline{M}_{0,6} = D(A_1) \\ \overline{M}_{0,4} \times \overline{M}_{0,7} \to \overline{M}_{0,4} \times [\overline{M}_{0,4} \times \overline{M}_{0,4}] = D(A_2), \end{cases}$$

where in the second case the map is induced (with appropriate ordering) from

$$\overline{M}_{0,7} \xrightarrow{\pi_{1,2,3,4} \times \pi_{4,5,6,7}} \overline{M}_{0,4} \times \overline{M}_{0,4}; \tag{10.9.1}$$

$$\text{for (10.5.3):} \quad q = \begin{cases} \overline{M}_{0,5} \times \overline{M}_{0,5} \times \overline{M}_{0,4} \to \overline{M}_{0,4} \times \overline{M}_{0,4} \times \overline{M}_{0,4} = D(A_2^{\times 3}) \\ \overline{M}_{0,5} \times \overline{M}_{0,5} \times \overline{M}_{0,4} \xrightarrow{\mathrm{pr}_1} \overline{M}_{0,5} = D(A_1) \cap D(A_1'). \end{cases}$$

$$\text{for (10.5.4):} \quad q : \ \overline{M}_{0,8} \to \overline{M}_{0,6} = D(A_1 \subset E_6).$$

10.10. Definition. We refer to the second possibility for $A_2 \subset E_7$ (resp. the second possibility for $A_3^{\times 2} \subset E_7$) as a *non-flat* $D(A_2)$ (resp. a *non-flat* $D(A_3 \times A_3)$).

Proof. By Th. 9.18 the map of fans determines the image of each boundary stratum, which in turn determines the pullback map on Weil divisors for the map of a stratum to its image (since by Th. 9.18, Prop. 10.2, and Prop. 10.5 the pullback of a boundary divisor for this map will be reduced). By Lemma 10.8 it is enough to check that this pull-back agrees with the pull-back for the maps in the theorem (which we denote by t). We use Prop. 9.17, 10.2, and 10.5 and suggest that the reader draws Dynkin diagrams. We use the standard $D_5 = E_5 \subset E_6 \subset E_7$ (as in Rk. 4.9) and $A_2^{\times 3} \subset E_6$, $A_3^{\times 2}, A_7 \subset E_7$ (as in Fig. 2) unless otherwise noted. Each paragraph has its own temporary notations for root subsystems.

(10.2.2), $A_1 \subset D_5$. We choose $A_1 = 7$ in D_5 with simple roots $7, 123, 23, 34, 45$. We identify $D(A_1) = M_{0,\{1,\ldots,6\}}$ and $t = \pi_{2,3,4,5}$. Up to conjugation, any boundary divisor D of $p(D(A_1))$ is its intersection with $D(D_2' = A_1'' \times A_1''' \subset D_5)$, where $A_1'' = 23$ and $A_1''' = 45$. The only components of $p^{-1}(D(D_2'))$ intersecting $D(A_1)$ are $D(A_1'')$, $D(A_1''')$, $D(A_2^{\times 3})$, and $D(A_2'^{\times 3})$, where $A_2 = \{123, 456, 7\}$, and $A_2' = \{145, 236, 7\}$. Thus $q^{-1}(D) = \delta_{23} \cup \delta_{45} \cup \delta_{123} \cup \delta_{145} = t^{-1}(\delta_{23})$.

(10.2.2), $A_1 \not\subset D_5$. We choose $A_1 = 7$. We have $t = \pi_{1,2,3,4,5}$. Up to conjugation, a boundary divisor D of $\overline{Y}(D_5)$ is $D(D_2 = A_1' \times A_1'') \subset \overline{Y}(D_5)$, where $A_1' = 45$, and $A_1'' = 123$. The only components of $p^{-1}(D)$ intersecting $D(A_1)$ are $D(A_1')$ and $D(A_2^{\times 3})$, where $A_2 = \{12, 13, 23\}$. Thus $q^{-1}(D(D_2)) = \delta_{123} \cup \delta_{45} = t^{-1}(\delta_{45})$.

(10.2.3). Up to conjugation any boundary divisor D of $p(D(A_2^{\times 3}))$ is its intersection with $D(D_2 = A_1 \times A_1' \subset D_5)$, where $A_1 = 12$ and $A_1' = 345$. The only component of $p^{-1}(D(D_2))$ intersecting $D(A_2^{\times 3})$ is $D(A_1)$. It follows that $q^{-1}(D) = \delta_{12} \times \overline{M}_{0,4} \times \overline{M}_{0,4} = t^{-1}(\delta_{12})$.

(10.5.2), $A_2 \cap E_6 = A_1$. Choose $A_2 = \{56, 57, 67\}$ and identify the $\overline{M}_{0,7}$ component of $D(A_2)$ with $\overline{M}_{0,\{0,\dots,4,x,y\}}$, $t = \pi_{1,\dots,4,x,y}$. $p(D(A_2)) = D(A_1 \subset E_6) = \overline{M}_{0,\{1,2,3,4,x,y\}}$ has 3 conjugacy classes of boundary divisors:

(a) the intersection with $D(A_1' \subset E_6)$, where $A_1' = 7$. The only components of $p^{-1}(D(A_1'))$ that intersect $D(A_2)$ are $D(A_7)$ and $D(A_3^{\times 2})$, where A_3 has simple roots $12, 23, 34$. It follows that $q^{-1}(D) = \overline{M}_{0,4} \times [\delta_{01234} \cup \delta_{1234}] = t^{-1}(\delta_{xy})$.

(b) the intersection with $D(A_1' \subset E_6)$, where $A_1' = 123$. The only two components of $p^{-1}(D(A_1'))$ that intersect $D(A_2 \subset E_7)$ are $D(A_1')$ and $D(A_2)$, where $A_2 = \{123, 4, 567\}$. It follows that $q^{-1}(D) = \overline{M}_{0,4} \times [\delta_{4x} \cup \delta_{04x}] = t^{-1}(\delta_{4x})$.

(c) the intersection with $D(A_2'^{\times 3} \subset E_6)$, where $A_2' = \{123, 7, 456\}$. The only two components of $p^{-1}(D(A_2'^{\times 3}))$ that intersect $D(A_2 \subset E_7)$ are $D(A_2'')$ and $D(A_3^{\times 2})$, where $A_2'' = \{12, 13, 23\}$. It follows that $q^{-1}(D) = \overline{M}_{0,4} \times [\delta_{0123} \cup \delta_{123}] = t^{-1}(\delta_{456})$.

(10.5.2), $A_2 \subset E_6$. Choose $A_2 = \{123, 456, 7\}$. Then $A_5 = A_2^{\perp}$ has simple roots 12, 23, 347, 45, 56. We have $D(A_2) = \overline{M}_{0,\{a,b,c,d\}} \times \overline{M}_{0,\{1,2,3,x,5,6,y\}}$, $t = id \times [\pi_{123y} \times \pi_{x56y}]$. Up to symmetries, $p(D(A_2)) = \overline{M}_{0,4}^{\times 3}$ has 2 types of boundary divisors:

(a) the intersection with $D(A_1)$ for $A_1 = 7$. The only component of $p^{-1}(D(A_1))$ that intersects $D(A_2)$ is $D(A_1)$. Thus $q^{-1}(D) = \delta_{c,d} \times \overline{M}_{0,7} = t^{-1}(\delta_{c,d} \times \overline{M}_{0,4} \times \overline{M}_{0,4})$.

(b) the intersection with $D(A_1)$ for $A_1 = 12$. Components of $p^{-1}(D(A_1))$ that intersect $D(A_2)$ are $D(A_1)$, $D(A_2^{(i)})$, $i = 4, 5, 6$, $D({}^{(i)}A_3^{\times 2})$, $i = 4, 5, 6$, and $D(A_7)$, where $A_2^{(i)} = \{12, 1i7, 2i7\}$, ${}^{(i)}A_3$ has simple roots $7, 123, 3i$, and A_7 has simple roots 12, 1, 7, 123, 34, 45, 56. The corresponding decomposition of $q^{-1}(D)$ is

$$\overline{M}_{0,4} \times [\delta_{12} \cup \delta_{12x} \cup \delta_{125} \cup \delta_{126} \cup \delta_{1256} \cup \delta_{12x6} \cup \delta_{12x5} \cup \delta_{12x56}] = t^{-1}(\overline{M}_{0,4} \times \delta_{12} \times \overline{M}_{0,4}).$$

(10.5.3), $A_3^{\times 2} \cap E_6 = A_2^{\times 2}$. $D(A_3^{\times 2}) = \overline{M}_{0,\{0,1,2,3,x\}} \times \overline{M}_{0,\{4,5,6,7,y\}} \times \overline{M}_{0,\{a,b,c,d\}}$. Then $t = \pi_{1,2,3,x} \times \pi_{4,5,6,y} \times id$. Up to symmetries, $p(D(A_3^{\times 2})) = \overline{M}_{0,4}^{\times 3}$ has the following types of boundary divisors:

(a) the intersection with $D(A_1)$ for $A_1 = 7$. The only component of $p^{-1}(D(A_1))$ that intersects $D(A_3^{\times 2})$ is $D(A_7)$. Thus $q^{-1}(D) = \overline{M}_{0,5}^{\times 2} \times \delta_{c,d} = t^{-1}(\overline{M}_{0,4}^{\times 2} \times \delta_{c,d})$.

(b) the intersection with $D(A_1)$ for $A_1 = 123$. The only component of $p^{-1}(D(A_1))$ that intersects $D(A_3^{\times 2})$ is $D(A_1)$. Thus $q^{-1}(D) = \overline{M}_{0,5}^{\times 2} \times \delta_{b,d} = t^{-1}(\overline{M}_{0,4}^{\times 2} \times \delta_{b,d})$.

(c) the intersection with $D(A_1)$ for $A_1 = 12$. Components of $p^{-1}(D(A_1))$ that intersect $D(A_3^{\times 2})$ are $D(A_1)$ and $D(A_2)$ with $A_2 = \{1, 12, 2\}$. It follows that $q^{-1}(D) = [\delta_{12} \cup \delta_{012}] \times \overline{M}_{0,5} \times \overline{M}_{0,4} = t^{-1}(\delta_{12} \times \overline{M}_{0,5} \times \overline{M}_{0,4})$.

(10.5.3), $A_3^{\times 2} \cap E_6 = A_3 \times A_1 \times A_1'$. We choose $A_3^{\times 2}$ containing A_3 with simple roots $12, 23, 34$. The $\overline{M}_{0,5}$ component of $D(A_3^{\times 2})$ that corresponds to A_3 is identified with $\overline{M}_{0,\{1,2,3,4,x\}}$ and its boundary divisors are identified with $A_1 \subset A_3$ ($ij \mapsto \delta_{ij}$) and $A_2 \subset A_3$ ($\{ij, jk, ik\} \mapsto \delta_{ijk}$). Since the link of $\sigma = \{A_1, A_1'\} \subset \mathcal{R}(E_6)$ is also equivariantly identified with $A_1, A_2 \subset A_3$ (with A_2 corresponding to $A_2 \times A_2^{\perp}$), the boundary divisors of $\overline{M}_{0,5} = D(A_1) \cap D(A_1) \subset \overline{Y}(E_6)$ correspond to $A_1, A_2 \subset A_3$ in the same way as above (by the argument used in Prop. 9.17). It remains to note that the only component of $p^{-1}(D(A_1))$ (resp. $p^{-1}(D(A_2^{\times 3}))$) for $A_1 \subset A_3$ (resp. $A_2 \subset A_3$) that intersects $D(A_3^{\times 2})$ is $D(A_1)$ (resp. $D(A_2)$). Therefore, $q^{-1}(D) = t^{-1}(D)$.

(10.5.4). There are 2 types of boundary divisors D of $p(D(A_7)) = \overline{M}_{0,6}$.

(a) the intersection with $D(A_1)$ for $A_1 = 12$. The components of $p^{-1}(D(A_1))$ that intersect $D(A_7)$ are $D(A_1)$, $D(A_2)$ with $A_2 = \{1, 12, 2\}$, $D(A_2')$ with $A_2' =$

$\{12,17,27\}$, and $D(A_3^{\times 2})$, where A_3 has simple roots $1,12,27$. It follows that $q^{-1}(D) = \delta_{12} \cup \delta_{012} \cup \delta_{127} \cup \delta_{0127} = t^{-1}(\delta_{12})$.

(b) the intersection with $D(A_2^{\times 3})$ for $A_2 = \{12,23,13\}$. The components of $p^{-1}(D(A_2^{\times 3}))$ that intersect $D(A_7)$ are $D(A_2)$, $D(A_3^{\times 2})$, where A_3 has simple roots $1,12,23$, $D(A_3'^{\times 2})$, where A_3' has simple roots $12,23,37$, and $D(A_2')$ with $A_2' = \{45,46,56\}$. It follows that $q^{-1}(D) = \delta_{123} \cup \delta_{0123} \cup \delta_{1237} \cup \delta_{456} = t^{-1}(\delta_{123})$. □

10.11. Lemma. *Suppose $A_1, A_1' \subset E_6$ are orthogonal. The codimension 2 stratum $Z := D(A_1) \cap D(A_1') \subset \overline{Y}(E_6)$ has trivial projectivized normal bundle.*

Proof. It suffices to prove that normal bundles to Z in $D(A_1)$ and $D(A_1')$ are isomorphic. It is well-known that, for the embedding $\overline{M}_{0,n} = \delta_{n,n+1} \subset \overline{M}_{0,n+1}$, $\mathcal{O}(\delta_{n,n+1})|_{\delta_{n,n+1}} = \psi_n^*$. Therefore it suffices to check that Z embeds both in $D(A_1)$ and $D(A_1')$ as a section δ_{xy}, where we identify both $D(A_1)$ and $D(A_1')$ with $\overline{M}_{0,\{1,2,3,4,x,y\}}$ and Z with $\overline{M}_{0,\{1,2,3,4,x\}}$ as in the previous proof (the case (10.5.3), $A_3^{\times 2} \cap E_6 = A_3 \times A_1 \times A_1'$). But this follows from the identification of the link of Z in $\mathcal{R}(E_6)$ with boundary divisors of $\overline{M}_{0,5}$ because other δ_{ij} divisors of $D(A_1)$ and $D(A_1')$ either properly intersect Z or are disjoint from it. □

10.12. Lemma. *Let $\pi : Y \to Y'$ be a dominant morphism from an integral scheme to a normal scheme, with reduced fibres of constant dimension. Then π is flat.*

Proof. Use [EGA4, 15.2.3] and [EGA4, 14.4.4]. □

10.13. For basic properties of log structures we refer to [Kat] and [Ol]. Any log structure we use in this paper will be toric, i.e., the space will come with an evident map to a toric variety, and we endow the space with the pullback of the toric log structure on the toric variety. In fact, we do not make any use of the log structure itself, only the bundles of log (and relative log) differentials as in [KT, 2.19].

If X is a toric variety with torus T and $Y \subset X$ is a closed subvariety, we refer to the multiplication map $T \times Y \to X$, $(t,y) \mapsto t \cdot y$ as the *structure map.*

10.14. Proposition. *Let $X \xrightarrow{\pi} X'$ be a dominant toric map of toric varieties with reduced fibres of constant dimension. Let $Y \subset X$, $Y' \subset X'$ be closed subvarieties with smooth structure maps. Let $W \subset Y$, $W' \subset Y'$ be irreducible strata. Suppose π induces dominant maps $Y \xrightarrow{p} Y'$ and $W \xrightarrow{q} W'$. If q is log smooth at $z \in W$ then p is log smooth at z.*

Proof. Suppose first that $\operatorname{codim}_Y W = \operatorname{codim}_{Y'} W' = 1$. We have a commutative diagram of vector bundles on W

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^1_W(\log) & \longrightarrow & \Omega^1_Y(\log)|_W & \xrightarrow{r} & \mathcal{O}_W & \longrightarrow & 0 \\
 & & {\scriptstyle q^*}\uparrow & & {\scriptstyle p^*}\uparrow & & \| & & \\
0 & \longrightarrow & q^*(\Omega^1_{W'}(\log)) & \longrightarrow & q^*(\Omega^1_{Y'}(\log)|_{W'}) & \xrightarrow{r} & q^*(\mathcal{O}_{W'}) & \longrightarrow & 0
\end{array}$$

Here $\Omega^1_Y(\log) = \Omega^1_Y(\log B_Y)$ and r is the residue map. The first vertical arrow is a subbundle embedding by assumption. Thus the second arrow is a subbundle embedding, i.e., $Y \to Y'$ is log smooth at $z \in W$.

Suppose now that $\operatorname{codim}_Y W = 1$ and $W' = Y'$. Then the map $p^*\Omega^1_{Y'}(\log B_{Y'}) \to \Omega^1_Y(\log B_Y)|_W$ factors through $\Omega^1_W(\log B_W)$ and so is a subbundle embedding.

The general case follows by induction: If $W' = Y'$ find a stratum $W_0 \supset W$ such that $\operatorname{codim}_Y W_0 = 1$. Then $W_0 \to Y'$ is log smooth, and we are done by induction on $\operatorname{codim}_Y W$. If $W' \neq Y'$, choose a component W_0 of $p^{-1}(W')$ containing W. Then $W_0 \to W'$ is log smooth. Choose a stratum $W_0' \supset W'$ such that $\operatorname{codim}_{W_0'} W' = 1$, and let W_1 be a component of $p^{-1}(W_0')$ such that $W_0 \subset W_1$ and $W_1 \to W_0'$ is dominant. Then $\operatorname{codim}_{W_1} W_0 = 1$, so $W_1 \to W_0'$ is log smooth and we are done by induction on $\operatorname{codim}_{Y'} W'$. □

10.15. Proposition. *Let $X \xrightarrow{\pi} X'$ be a flat map of toric varieties with reduced fibers extending a surjective homomorphism $T \to T'$ of algebraic tori, with kernel T''. Assume X' is smooth. Let $Y \subset X$, $Y' \subset X'$ be subvarieties with smooth structure maps and assume $\pi|_Y$ is a dominant map $Y \xrightarrow{p} Y'$, log smooth at $y \in Y$. Then the induced map $f : Y \times T'' \to F := Y' \times_{X'} X$ is smooth at y.*

Proof. Define Q (locally near y) by the exact sequence

$$0 \to \Omega^1_{Y' \times T'/X'}|_{Y \times T''} \to \Omega^1_{Y \times T/X}|_{Y \times T''} \to Q \to 0$$

The first term is the pullback of the log cotangent bundle of Y' and the second of the log cotangent bundle of Y [KT, 2.19]. Thus Q is a vector bundle of rank

$$r = \dim Y - \dim Y' = \dim(Y \times T'') - \dim F.$$

A simple computation shows that $Q = \Omega^1_{Y \times T''/F}$. Therefore f is equidimensional and has smooth fibers (locally near y). It suffices to prove that F is normal: then f is flat by Lemma 10.12, and so smooth [EGA4, 6.8.6] (locally near y).

Observe $F \to Y'$ is flat, has reduced Cohen–Macaulay fibres, and normal generic fibre. Since Y' is smooth, F is Cohen–Macaulay and has R_1. Thus F is normal. □

We recall the definition of the Kollár–Shepherd-Barron–Alexeev moduli stack $\overline{\mathcal{M}}$ of stable surfaces with boundary. Let k be an algebraically closed field of characteristic zero.

10.16. Definition. We define the Kollár–Shepherd-Barron–Alexeev moduli stack $\overline{\mathcal{M}}$ of stable surfaces with boundary over k as follows. For a scheme T/k, the objects of $\overline{\mathcal{M}}(T)$ are families $(\mathcal{S}, \mathcal{B} = \mathcal{B}_1 + \cdots + \mathcal{B}_n)/T$, where $\mathcal{S}/T$ is a flat family of surfaces and each $\mathcal{B}_i/T$ is a flat family of codimension one subschemes of $\mathcal{S}/T$, satisfying the following properties.

(1) For every geometric fibre (S, B) of $(\mathcal{S}, \mathcal{B})/T$, the surface S is reduced and satisfies Serre's S_2 condition, and the codimension one subscheme $B \subset S$ is reduced. The pair (S, B) has semi log canonical singularities [Al5, 1.4] and the $\mathbb{Q}$-line bundle $\omega_S(B)$ is ample.
(2) Let $j : \mathcal{S}^0 \subset \mathcal{S}$ denote the open locus where $\mathcal{S}/T$ is Gorenstein and $\mathcal{B}_i \subset \mathcal{S}$ is Cartier for each i. (Then the complement of $\mathcal{S}^0$ has finite fibres by (1).) Define
$$\omega_{\mathcal{S}/T}(\mathcal{B})^{[m]} := j_*(\omega_{\mathcal{S}^0/T}(\mathcal{B}^0)^{\otimes m}).$$
Then the formation of $\omega_{\mathcal{S}/T}(\mathcal{B})^{[m]}$ commutes with base change for all $m \in \mathbb{Z}$.

10.17. Theorem. [Ko6] *The stack $\overline{\mathcal{M}}$ is a separated Deligne–Mumford stack and its connected components are of finite type. Let $\overline{\mathcal{M}}' \subset \overline{\mathcal{M}}$ denote the closure of the locus of pairs (S, B) such that S is normal. Then $\overline{\mathcal{M}}'$ is proper.*

We consider smooth del Pezzo surfaces S of degree $9 - n$, $4 \leq n \leq 8$, with boundary B the sum of the (-1)-curves. Then $K_S + B$ is ample and the pair (S, B) is log canonical if B is a normal crossing divisor. A marking of S corresponds to a labelling of the (-1)-curves.

10.18. Definition. Let $B_p \subset \overline{Y}(\Delta)$ be the union of the boundary divisors which surject onto $\overline{Y}(\Delta')$ — thus for D_n these are the divisors $D(D_2)$ for $D_2 \not\subset D_{n-1}$, and for E_n the divisors $D(A_1)$ for $A_1 \not\subset E_{n-1}$. We call them *horizontal divisors*.

Let $\overline{M}$ denote the coarse moduli space of $\overline{\mathcal{M}}$.

10.19. Theorem. *For $\Delta = D_n$ or E_n, $n \leq 6$, $p : (\overline{Y}(\Delta), B_p) \to \overline{Y}(\Delta')$ is a flat log smooth family of stable pairs. It induces an isomorphism $\overline{Y}(\Delta') \xrightarrow{\sim} \overline{M}_{0,n-1}$ for D_n and a closed embedding $\overline{Y}(\Delta') \subset \overline{M}$ for E_n (assuming* $\operatorname{char} k = 0$*).*

Proof. This is well-known for D_n: p is the universal family of stable rational curves. For $\Delta = E_5$, $\overline{Y}(\Delta') = \{pt\}$ and the claim is easy. So it is enough to consider $\Delta = E_6$.

The claims are clear over the interior $Y(E_5)$ — it is well known that the -1 curves on a del Pezzo of degree at least 4 have normal crossings. For each boundary divisor $D \subset \overline{Y}(E_6)$, the restriction $p|_D$ is described by Prop. 10.9. It has reduced fibers and is flat of relative dimension ≤ 2 and log smooth (instances of the D_n case). Flatness of p follows from Lemma 10.12 and log smoothness from Prop. 10.14.

Let T_π be the kernel of $\pi : T_{\overline{Y}(E_6)} \to T_{\overline{Y}(E_5)}$. Let $F := \overline{Y}(E_5) \times_{X(\mathcal{F}(E_5))} X(\mathcal{F}(E_6))$. By Prop. 10.15, $T_\pi \times \overline{Y}(E_6) \to F$ is smooth over $\overline{Y}(E_5)$. It follows that fibers of p have stable toric singularities. By Cor. 9.2, $K_{\overline{Y}(E_6)} + B$ is ample. Since $B = p^*(B) + B_p$, $K_{\overline{Y}(E_6)} + B$ restricts to the log canonical bundle of each fibre, and thus p is a family of stable pairs. Consider the corresponding map $\overline{Y}(E_5) \to \overline{M}$. The closed embedding (9.18.1) factors through $\overline{M}$ by Lemma 10.20. Thus $\overline{Y}(E_5) \to \overline{M}$ is a closed embedding. □

10.20. Lemma. *Assume* $\operatorname{char} k = 0$ *and $n \leq 6$. The KSBA cross-ratios are regular in a neighbourhood of the closure of the locus of smooth marked del Pezzo surfaces of degree $(9-n)$ in the moduli stack $\overline{\mathcal{M}}$.*

Proof. Recall that, for a smooth del Pezzo surface with normal crossing boundary, a KSBA cross-ratio is the cross-ratio of 4 points on a (-1)-curve cut out by 4 other (-1)-curves. Consider the universal family $(\mathbb{S}, \mathbb{B}) \to \overline{\mathcal{M}}$. Let (S, B) be a fibre over the closure of the locus of smooth marked del Pezzo surfaces. We claim that $(B_i, \sum_{j \neq i} B_j|_{B_i})$ is a stable pointed curve of genus 0 for each i. First, B_i is a nodal curve and the marked points are smooth and distinct because (S, B) has semi log canonical singularities. Second, (S, B) has stable toric singularities by the first clause of Thms. 10.19 and 10.31, so, in particular, $\omega_S(B)$ is invertible. It follows that the adjunction formula $\omega_{B_i}(\sum_{j \neq i} B_j|_{B_i}) = \omega_S(B)|_{B_i}$ holds [Ko2, 16.4.2]. So $\omega_{B_i}(\sum_{j \neq i} B_j|_{B_i})$ is ample and $(B_i, \sum_{j \neq i} B_j|_{B_i})$ is stable, as required. Because stable curves deform to stable curves, the same is true for nearby fibres. We obtain a map from a neighbourhood of the closure of the locus of smooth del Pezzo surfaces in $\overline{\mathcal{M}}$ to $\overline{M}_{0,m}$, where m is the number of marked points on B_i. Forgetting all but 4 of the marked points we obtain maps to $\overline{M}_{0,4}$ extending the KSBA cross-ratios on the locus of smooth del Pezzo surfaces. □

10.21. Definition. Let $\tilde{Y}(E_n) \subset X(\tilde{\mathcal{F}}(E_n))$ be the closure of $Y(E_n)$, $n = 6$ or 7. Let $\tilde{p} : \tilde{Y}(E_7) \to \tilde{Y}(E_6)$ denote the restriction of $\tilde{\pi} : X(\tilde{\mathcal{F}}(E_7)) \to X(\tilde{\mathcal{F}}(E_6))$. An *Eckhart point* of $\overline{Y}(E_7)$ (resp. $\tilde{Y}(E_7)$) is a point on the intersection of three horizontal $D(A_1)$ divisors (resp. their strict transforms).

10.22. Proposition. *$(\overline{Y}(E_7), B_p) \xrightarrow{p} \overline{Y}(E_6)$ has the following properties: it is flat with reduced fibers outside of the union D of non-flat $D(A_3^{\times 2})$ divisors; log smooth outside of D and Eckhart points; at an Eckhart point z away from D, p is smooth and B_p restricts on the fibre to 3 pairwise transversal curves intersecting at z.*

Proof. $W(E_6)$ acts transitively on Fano simplices of $\mathcal{R}(E_7)$ (since E_6 contains at most 4 orthogonal A_1's, $E_7 \setminus E_6$ at most 3, and 4 orthogonal A_1's are contained in a unique Fano simplex). This implies the result on the purely A_1 locus in $\overline{Y}(E_7)$, as each fibre (S, B) of $p|_V$, for a chart V of Lemma 9.4, is an open subset of $\mathbb{P}^2$ of Fig. 6 with the lines through one of the points as the boundary.

By Prop. 10.9, $p|_{D(\Gamma)}$ is flat of relative dimension ≤ 2, has reduced fibers, and is log smooth for $\Gamma = A_7$, a flat $A_3 \times A_3$, or a flat A_2. By Prop. 10.14, it suffices to prove that the map $p|_{D(A_2)}$ has properties of Prop. 10.22 for any $A_2 \subset E_6 \subset E_7$. By Prop. 10.9, it suffices to prove this for the map (10.9.1), which we denote by q.

$q|_{M_{0,7}}$ is smooth, and thus by Prop. 10.14 we just have to consider how q restricts to the possible boundary divisors of $\overline{M}_{0,7}$. One checks (by running through the list of possibilities) that q restricts to a flat log smooth map onto a stratum of $\overline{M}_{0,4}^{\times 2}$ except for the boundary divisors given by subsets of form $\{1,5,4\}$, or $\{1,5\}$ (for example, $\{1,4\}$ gives the boundary divisor $\overline{M}_{0,\{2,\dots,7\}}$ and q restricts to the canonical projection $\pi_{\{4,5,6,7\}}$). It is easy to check using Prop. 9.17 that divisors of type $\{1,5,4\}$ (resp. of type $\{1,5\}$) are precisely restrictions of non-flat $D(A_3 \times A_3)$ divisors (resp. vertical $D(A_1)$ divisors). Consider the open set $U \subset \overline{M}_{0,7}$, the complement to all the boundary divisors not of the type $\{1,5\}$. It is enough to prove the claim for $q|_U$. Note U has 9 boundary divisors, $\{i,j\}$ with $i < 4$, $j > 4$, and that two meet iff the subsets are disjoint. On their intersection, say $\delta_{1,5} \cap \delta_{2,6}$,

$$q|_{\delta_{1,5}\cap\delta_{2,6}} : \overline{M}_{0,\{1,2,3,4,7\}} \xrightarrow{\pi_{1,2,3,4}\times\pi_{1,2,4,7}} \overline{M}_{0,4} \times \overline{M}_{0,4}$$

is the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at three points along the diagonal. The exceptional divisors are outside of U. So the map is an isomorphism, i.e., the intersection of any two boundary divisors gives a section. Thus q is smooth in a neighborhood of any codimension two stratum of U. The restriction to the interior of any boundary divisor is easily seen to be smooth. Further, each codimension three stratum is carried isomorphically to the diagonal of $\overline{M}_{0,4} \times \overline{M}_{0,4}$ (which is not a stratum). It follows that $q|_U$ is flat, and log smooth outside of the codimension three strata. □

10.23. Proposition. *$\tilde{p} : \tilde{Y}(E_7) \to \tilde{Y}(E_6)$ is flat, has reduced fibers, and is log smooth outside of Eckhart points. At Eckhart points the map is smooth, and B_p restricts on the fibre to three pairwise transversal curves intersecting at a point.*

Proof. By Prop. 10.22 p is flat away from the non-flat $D(A_3 \times A_3)$ divisors. It follows that on this locus, $\tilde{p}$ is just the pullback [T, 2.9]. Thus (as log smoothness and reduced fibers are preserved by pullback) Prop. 10.22 implies that $\tilde{p}$ has the required properties outside the inverse image of non-flat $D(A_3 \times A_3)$ divisors.

Fix a non-flat divisor $D(\Theta)$, where $\Theta = A_3 \times A_3'$, $A_3 \subset E_6$. Let $Z \subset \overline{Y}(E_6)$ be its image, $Z = D(A_1^{(1)}) \cap D(A_1^{(2)}) \simeq \overline{M}_{0,5}$. By Prop. 10.9, $D(\Theta) = \overline{M}_{0,5}(A_3) \times \overline{M}_{0,5}(A_3') \times \overline{M}_{0,4}$ and p restricts to the projection onto the first factor.

10.24. Lemma. *$p^{-1}D(A_1^{(i)})$ near $D(\Theta)$ has irreducible components $D(\Theta)$, $D(A_1^{(i)})$, $D(A_7^{(i)})$, $D(A_2^{(i)})$, $D(\tilde{A}_2^{(i)})$: take all A_7's containing Θ and all A_2's contained in A_3'. For any of these divisors $D \neq D(\Theta)$, the map $p|_D$ is log smooth and flat near $D(\Theta)$. The intersection of $D(\Theta)$ with $D(A_7^{(i)})$ (resp. with one of $D(A_1^{(i)})$, $D(A_2^{(i)})$, $D(\tilde{A}_2^{(i)})$) is pulled back from a point of the $\overline{M}_{0,4}$ component (resp. from a divisor of the $\overline{M}_{0,5}(A_3')$ component) of $D(\Theta)$ according to the following picture, where we realize $\overline{M}_{0,5}$ as the blow up of $\mathbb{P}^2$ in 4 points (corresponding to A_2 divisors). Let Q be the union of codimension 2 strata in $p^{-1}(Z)$ (near $D(\Theta)$) not contained in $D(\Theta)$. Dotted lines illustrate $D(\Theta) \cap Q$.*

Proof. The description of $p^{-1}(D(A_1^{(i)}))$ follows from Prop. 10.5. Intersections of divisors are computed in Prop. 9.17. By Prop. 10.9 we only have to prove that $p|_{D(A_1^{(i)})}$ is log smooth and flat. But $D(A_1^{(i)}) \cap D(\Theta) \simeq \overline{M}_{0,5} \times \overline{M}_{0,4} \times \overline{M}_{0,4}$ and $p|_{D(A_1^{(i)})\cap D(\Theta)} \simeq \mathrm{pr}_1$. Clearly log smooth and flat of relative dimension 2. Thus

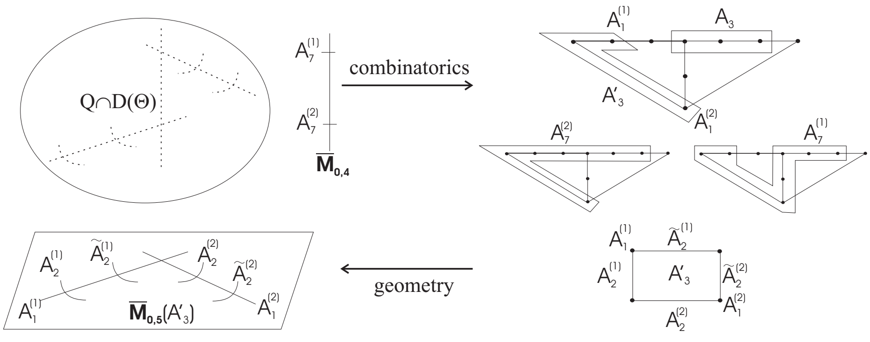


FIGURE 8.

$p|_{D(A_1^{(i)})}$ is log smooth by Prop. 10.14 and flat by Lemma 10.12 (since p is flat outside $D(\Theta)$). ☐

By Prop. 10.5 and Th. 9.18 we have a commutative diagram

$$\begin{array}{ccccc} \tilde{Y}(E_7) & \longrightarrow & \hat{Y}(E_7) & \xrightarrow{b_7} & \overline{Y}(E_7) \\ {\scriptstyle\tilde{p}}\downarrow & & {\scriptstyle\hat{p}}\downarrow & & {\scriptstyle p}\downarrow \\ \tilde{Y}(E_6) & \longrightarrow & \hat{Y}(E_6) & \xrightarrow{b_6} & \overline{Y}(E_6) \end{array}$$

where b_6 (resp. b_7) is the blowup of Z (resp. of $W := p^{-1}(Z)$). For any divisor D on $\overline{Y}(E_6)$ or $\overline{Y}(E_7)$, we denote by $\hat{D}$ its strict transform. Prop. 10.23 follows from an analogous statement for $\hat{p}$, which is proved in Lemma 10.27 below. ☐

10.25. REMARK. It is clear from Fig. 8 and transversality that $I_W = I_{D(\Theta)} \cdot I_Q$ and Q is regularly embedded, near $D(\Theta)$. Thus blowing up W is the same as blowing up Q and $\hat{D}(\Theta) = b_7^{-1}(D(\Theta))$ as $D(\Theta)$ does not contain any component of Q. Note that, at the generic point of intersection of irreducible components of Q, the blow-up is locally a product of $\mathbb{A}^3$ with S, the blow up of the union of two lines in $\mathbb{A}^3$ through the origin. Note that S has an ordinary double point.

10.26. LEMMA. *Suppose that $D \subset \partial\overline{Y}(E_7)$ surjects onto $D(A_1^{(i)}) \subset \overline{Y}(E_6)$. Then, near $D(\Theta)$, $\hat{D} \simeq D$, $\hat{D}(A_1^{(i)}) \simeq D(A_1^{(i)})$, and $\hat{D} \to \hat{D}(A_1^{(i)})$ is flat and log smooth.*

Proof. We blow up a divisor of $D(A_1^{(i)})$, so $\hat{D}(A_1^{(i)}) \to D(A_1^{(i)})$ is an isomorphism. It is clear from Fig. 8 that the scheme-theoretic intersection $Q \cap D$ is a divisor of D, so $\hat{D} \to D$ is an isomorphism. The last remark now follows from Lemma 10.24. ☐

A_3' contains 6 A_1's, $A_1^{(1)}, \ldots, A_1^{(6)}$, divided into three orthogonal pairs. One pair is $A_1^{(1)}, A_1^{(2)}$. Taking the other two pairs, and adding $\Theta^\perp$ to each, gives two triples of orthogonal A_1's in $E_7 \setminus E_6$, i.e., Eckhart triples. Note $\hat{D}(A_1) = b_7^{-1}(D(A_1))$ for any $A_1 \not\subset E_6$, since $D(A_1)$ does not contain any component of Q (because $D(A_1) \to \overline{Y}(E_6)$ is flat away from non-flat $D(A_3^{\times 2})$'s by Prop. 10.22).

10.27. LEMMA. *Along $\hat{D}(\Theta)$, $\hat{p}$ is flat, and log smooth away from the strata $\hat{D}(A_1) \cap \hat{D}(A_1') \cap \hat{D}(\Theta^\perp)$ for the two Eckhart triples above. Near an Eckhart triple $\hat{p}$ is smooth, and each of the three $\hat{D}(A_1)$'s from the triple meets the fibre in a smooth curve. These curves are pairwise transversal and intersect at a point.*

Proof. By Lemma 10.11, the exceptional divisors are products $E_{b_6} = Z \times \mathbb{P}^1$ and $E_{b_7} = Q \times \mathbb{P}^1$. By Lemma 10.24, for a boundary divisor D mapping onto $D(A_1^{(i)})$, the restriction $p|_D$ is flat and log smooth near $D(\Theta)$. In particular each component of Q is then flat and log smooth over its image. And thus the same holds for irreducible components of E_{b_7}, since $E_{b_7} \to E_{b_6}$ is a pullback of $Q \to Z$ (indeed just product with $\mathbb{P}^1$). Now by Prop. 10.14, log smoothness holds around E_{b_7}. Moreover, by the previous Lemma, we have log smoothness around any $\hat{D}$.

So it's enough to work outside of E_{b_7}, where b_7 is the identity, and away from any of the divisors D of Lemma 10.24. On this open subset $D(\Theta)$ is isomorphic to $Z \times U \times V$, where $U \subset \mathbb{P}^2$ is the complement of lines L_i, $i = 1, 2$ (that correspond to $A_1^{(i)}$ on Fig. 8) and $V = \mathbb{P}^1 \setminus \{P_1, P_2\}$ (where P_i corresponds to $A_7^{(i)}$).

Now we can describe the map $\tilde{p} : D(\Theta) \to E_{b_6} = Z \times \mathbb{P}^1$ (on the open set on which we are working). As it is the same over each point of $Z = \overline{M}_{0,5}$, we describe it as a rational map $f : \mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$. Since the projective normal bundle of Z is trivial, it is the rational map given by the linear system spanned by the two divisors $p^{-1}(D(A_1^{(i)})) - D(\Theta)$, $i = 1, 2$, restricted on $D(\Theta)$. On $\mathbb{P}^2 \times \mathbb{P}^1$ these divisors are $(L_i \times \mathbb{P}^1) \cup (\mathbb{P}^2 \times P_i)$. Thus in coordinates f is given by

$$f : ((X : Y : Z),\ (A : B)) \to (XA : YB)$$

where the lines L_i are given by $X = 0$, $Y = 0$, and the points P_i are are given by $A = 0$, $B = 0$. On $U \times V$ this map is clearly smooth, of relative dimension 2. This proves flatness. For log smoothness, we consider how the boundary divisors meet the fibres. There are 5 boundary divisors that meet $U \times V$, namely $D(A_1^{(i)})$ for $i = 3, 4, 5, 6$, and $D(\Theta^\perp)$. On Fig. 8, the first 4 divisors project on lines in $\mathbb{P}^2$ pairwise connecting 4 blown up points, and the last is the inverse image of a point from $\mathbb{P}^1$, corresponding to $D(\Theta^\perp)$. Clearly the picture on each fibre of f is of smooth divisors on a smooth surface that either intersect transversally or three of them meet at a point and are pairwise transversal. The triple points come from the orthogonal pairs of $A_1^{(i)}$, intersecting with $D(\Theta^\perp)$, which are exactly the Eckhart triples mentioned above. □

10.28. Lemma. *Let P be product of smooth (not necessarily proper) curves of log general type. Let $S \to P$ be a map with S smooth, which has a generically injective derivative. Then S is of log general type.*

Proof. We may replace P by $\dim S$ factors and assume the map is generically étale. If C is a curve of log general type then Ω^1_C is globally generated by log 1-forms. It follows that ω_P is globally generated by log canonical forms (by wedging), so the result now follows by pulling back forms. □

10.29. Proposition. *Let $Z = D(A_1) \cap D(A_1') \subset \overline{Y}(E_6)$ be a codimension 2 stratum. Then there is a KSBA cross-ratio g such that near Z we have $(g) = D(A_1) - D(A_2)$.*

Proof. Let M denote the lattice spanned by D_4 and let λ, ρ, μ, ν be the characters corresponding to simple roots $\varepsilon_1 - \varepsilon_2$, $\varepsilon_2 - \varepsilon_3$, $\varepsilon_3 - \varepsilon_4$, $\varepsilon_3 + \varepsilon_4$. Let $T = \mathrm{Hom}(M, \mathbb{G}_m)$. For $\alpha \in D_4^+$ let H_α denote the hypertorus $(\chi^\alpha = 1) \subset T$. Cayley [C] found an explicit family of smooth cubic surfaces over $T \setminus \bigcup H_\alpha$, see also [N, 5.1]:

$$\begin{gathered}\rho W(\lambda X^2 + \mu Y^2 + \nu Z^2 + (\rho - 1)^2(\lambda\mu\nu\rho - 1)^2 W^2 \\ +(\mu\nu + 1)YZ + (\lambda\nu + 1)XZ + (\lambda\mu + 1)XY \\ -(\rho - 1)(\lambda\mu\nu\rho - 1)W((\lambda + 1)X + (\mu + 1)Y + (\nu + 1)Z)) + XYZ = 0.\end{gathered} \tag{10.29.1}$$

The marking of the family (10.29.1) is given by labelling the tritangent planes of the fibres [N, Table 1, p. 10]. By [N, Prop. 7.1], the morphism $T \setminus \bigcup H_\alpha \to Y(E_6)$ defined by the family (10.29.1) is an open embedding.

10.30. Lemma. *The character χ^α for $\alpha \in D_4$ is a KSBA cross-ratio.*

Proof. We use the equations of the tritangent planes for the family (10.29.1) given in [N, Table 1, p. 10]. A fibre of the family contains the line $l = (W = X = 0)$, and has tritangent planes $(W = 0)$, $(X = 0)$ and $(X + \rho(\mu - 1)(\nu - 1)W = 0)$ through l. These give line pairs intersecting l in the points $Z/Y = \{0, \infty\}$, $\{-\mu, \frac{-1}{\nu}\}$, and $\{-1, \frac{-\mu}{\nu}\}$ respectively. Thus $\mu = \varepsilon_3 - \varepsilon_4$ is a KSBA cross-ratio (the cross-ratio of $0, -1, -\mu, \infty$). By $W(D_4)$-equivariance, the same is true for any $\alpha \in D_4$. □

We recall the construction of $\overline{Y}(E_6)$ from [N, p. 21–24]. Let Σ be the fan of Weyl chambers in $N_{\mathbb{R}}$ given by the Coxeter arrangement of root hyperplanes in D_4. Let $X = X(\Sigma)$ be the corresponding T-toric variety. Let $\overline{H}_\alpha \subset X$ be the closure of H_α. X is smooth and the arrangement of $\overline{H}_\alpha$'s at the identity $e \in T$ is locally isomorphic to the Coxeter arrangement of hyperplanes in $N \otimes k$. Let $\tilde{X} \to X$ be the blowup of this arrangement which corresponds to the "wonderful blowup" of the Coxeter arrangement. That is, we blowup the strata corresponding to irreducible root subsystems $\Gamma \subset D_4$ in order of increasing dimension. Explicitly, we blowup the identity $e \in T$, 12 curves, and 16 surfaces, which correspond to sub root systems of type D_4, A_3, and A_2 respectively. Following Naruki, we call these strata A_3-curves and A_2-surfaces. Then the KSBA cross-ratios of type I define a morphism $\tilde{X} \to \overline{Y}(E_6)$ which contracts the strict transforms of the exceptional divisors over the A_3-curves to A_1^2-strata in $\overline{Y}(E_6)$ and is an isomorphism elsewhere.

Let $A_3 = \{\varepsilon_i - \varepsilon_j\} \subset D_4$ and let $\Gamma \subset X$ be the corresponding A_3-curve. Γ meets the toric boundary in two points, which are interior points of the toric boundary divisors Δ_1 and Δ_2 corresponding to the positive and negative rays of the line $A_3^\perp = \mathbb{R} \cdot (1, 1, 1, 1)$ in the fan of X. The primitive generators of these rays in $N = M^\vee$ are $\pm\frac{1}{2}(1, 1, 1, 1)$. So, if $\alpha = \varepsilon_1 + \varepsilon_2 \in D_4$ then the divisor (χ^α) equals $\Delta_1 - \Delta_2$ plus some other toric boundary divisors. On the Naruki space Δ_1 and Δ_2 become A_1-divisors Y_1 and Y_2 meeting in a A_1^2 stratum Z, and $g = \chi^\alpha$ is a cross-ratio with the desired properties. □

10.31. THEOREM. *Let $\ddot{Y}(E_7) \to \tilde{Y}(E_7)$ be the blowup of the union of Eckhart points. The map $(\ddot{Y}(E_7), B_{\ddot{p}}) \xrightarrow{\ddot{p}} \tilde{Y}(E_6)$ is a flat family of stable pairs with stable toric singularities. For* $\operatorname{char} k = 0$*, the induced map $\tilde{Y}(E_6) \to \overline{M}$ is a closed embedding, with image the closure of the locus of pairs consisting of a smooth cubic surface without Eckhart points with boundary its 27 lines. The product of all KSBA cross-ratio maps embeds $\tilde{Y}(E_6)$ in the product of $\mathbb{P}^1$'s.*

Proof. By Prop. 10.23, the union of Eckhart points is a disjoint union of smooth codimension three strata contained in the smooth locus of $\tilde{p}$. Fibres of $\ddot{p}$ have stable toric singularities by Prop. 10.23, Th. 9.18 and Prop. 10.15. Let (F, B) be a fibre, and $(\tilde{F}, B)$ the corresponding fibre of $\tilde{p}$. The inverse image of an Eckhart point of $(\tilde{F}, B)$ is $\mathbb{P}^2$ with 3 general lines on it, glued to the rest of F along the fourth general line. Thus $K_F + B$ restricts to an ample divisor on the exceptional locus for $F \to \tilde{F}$. To prove $K_F + B$ is ample, by Th. 9.1 it is enough to show that any open stratum S of $(\tilde{F}, B)$ is log minimal. Let $\tilde{G} \subset X(\tilde{\mathcal{F}}(E_7))$ be the fibre of $\tilde{\pi}$ containing $\tilde{F}$. The structure map $T_\pi \times \tilde{F} \to \tilde{G}$ is smooth outside of Eckhart points, and the stratification of $\tilde{F}$ is induced from the stratification of $\tilde{G}$ by restriction. Strata of $\tilde{G}$ are orbits for T_π, thus in particular, S is smooth. Since $X(\tilde{\mathcal{F}}(E_7))$ embeds in the normalization of $X(\tilde{\mathcal{F}}(E_6)) \times_{X(\mathcal{F}(E_6))} X(\mathcal{F}(E_7))$, $\tilde{F}$ maps finitely to a fibre of p, and S has a quasi-finite map to an open stratum of $\overline{Y}(E_7)$. Thus S has a quasi-finite map to a product of $M_{0,4}$'s by Th. 9.18, and so is log minimal by Lemma 10.28. Thus $\ddot{p}$ is a flat family of stable pairs. Let $\tilde{Y}(E_6) \to \overline{M}$ be the induced map.

By Lemma 10.20, the KSBA cross-ratio maps are regular on $\overline{\mathcal{M}}$, and therefore on $\tilde{Y}(E_6)$. Thus it suffices to prove that the product of KSBA cross-ratio maps embeds $\tilde{Y}(E_6)$. By Th. 9.18 the cross-ratios of type I define an embedding of $\overline{Y}(E_6)$. So to prove that all cross-ratios embed $\tilde{Y}(E_6)$ we can work locally over $\overline{Y}(E_6)$. By Th. 9.18, and because an analogous statement holds for the map of toric varieties $X(\tilde{F}(E_6)) \to X(\mathcal{F}(E_6))$, $\tilde{Y}(E_6) \to \overline{Y}(E_6)$ is the blow up of all purely A_1 strata in order of increasing dimension. Let $P \in \overline{Y}(E_6)$ be a point lying on exactly m $D(A_1)$-divisors and write locally $(P \in \overline{Y}(E_6)) = (0 \in \mathbb{A}^4_{x_1,\dots,x_4})$, where $(x_1 = 0), \dots, (x_m = 0)$ are the $D(A_1)$-divisors through P. An easy toric calculation shows that the rational functions x_i/x_j, $1 \le i < j \le m$, embed $\tilde{Y}(E_6)$ in $\overline{Y}(E_6) \times (\mathbb{P}^1)^m$ (locally near P)). On the other hand, Prop. 10.29 shows that these rational functions correspond (locally) to KSBA cross-ratio maps on $\overline{Y}(E_6)$. □

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