AGNES

OPEN PROBLEMS

Moduli of Surfaces

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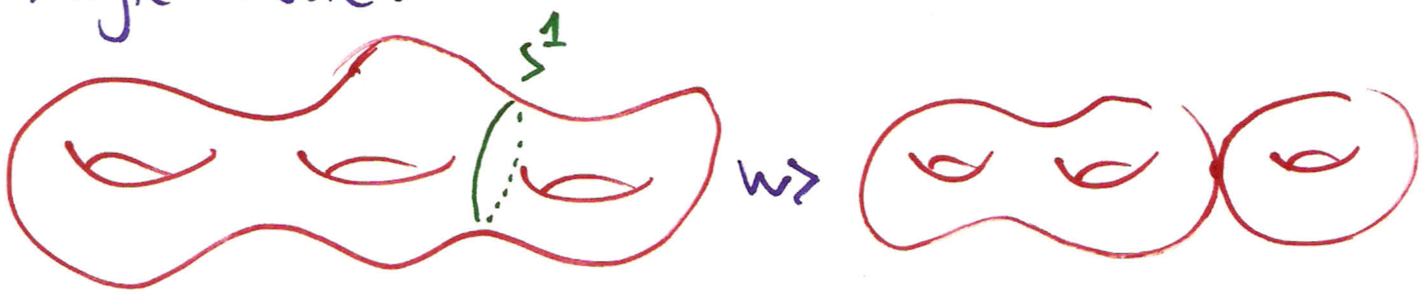
Curves

$\mathcal{M}_g :=$ moduli space parametrizing complex curves (Riemann surfaces) of genus $g \geq 2$

Riemann : $\mathcal{M}_g$ is complex manifold (orbifold) of dimension $3g-3$.

Deligne-Mumford : Compactification $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$ parametrizing stable curves (• allow nodes $(xy=0) \subset \mathbb{C}^2$ • require $|\mathrm{Aut}\, X| < \infty$)

Boundary $\partial \overline{\mathcal{M}}_g := \overline{\mathcal{M}}_g \setminus \mathcal{M}_g$ is union of components corresponding to topological types of degenerations w/ single node.



Surfaces

$\mathcal{M}$:= moduli space of surfaces of general type
(fix $c_1^2 = K_X^2$, $c_2 = e(X)$).

WARNING: $\mathcal{M}$ may be highly singular in general.

Kollár-Shepherd-Barron : Compactification
$\mathcal{M} \subset \overline{\mathcal{M}}$ parametrizing "stable surfaces"
(use 3-fold birational geometry to compactify family over punctured disc.) $\leftarrow$ Mori's MMP.

$\boxed{\text{Q1}}$ What are the components of $\partial\overline{\mathcal{M}} := \overline{\mathcal{M}} \setminus \mathcal{M}$?

Ex 1. Lefschetz degeneration.

$x^2+y^2+z^2=t$.

Wahl singularities (J. Wahl 1981)

$$X = (xy = z^n) \subset \mathbb{C}^3/_{\mathbb{Z}/n\mathbb{Z}}$$

$$\zeta : (x,y,z) \mapsto (\zeta x, \zeta^{-1} y, \zeta^a z), \quad (a,n) = 1.$$

Smoothing $\mathcal{X} = (xy = z^n + t) \subset \mathbb{C}^3/_{\mathbb{Z}/n\mathbb{Z}} \times \Delta$

Note: This is codimension 1 degeneration if we define $\mathcal{M}$ correctly (require $K^2_{X_t}$ constant)

I expect for "typical" component $\mathcal{M}$, this is it.

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Wahl singularities have Milnor fibre a rational homology ball (no "vanishing cycles").

$\boxed{Q2}$ How can we predict existence of Wahl degenerations in terms of <u>topology</u> of smooth surface?

Vector bundles

<u>A partial result</u>: Suppose $\pi_1(Y) = 0$, $H^{2,0}(Y) = 0$,

$Y \rightsquigarrow X$ Wahl degeneration.

Then $\exists$ exceptional vector bundle E on Y, $\mathrm{rk} = n$,

analogous to vanishing cycle.

$\mathrm{End}\, E = \mathbb{C}$

$H^i(\mathrm{End}\, E) = 0$, $i > 0$.

$\Rightarrow$ indecomposable, rigid, unobstructed.

$\boxed{Q3}$ What is classification of exceptional vector bundles on surface Y of general type?

(More generally, what are possible Chern classes of stable bundles?)

Gieseker: r, c_1 fixed, $c_2 \gg 0$

$\Rightarrow \exists E$ stable.

BUT: exceptional $\sim$ c_2 "minimal"

$$\chi(\mathrm{End}\, E) \overset{RR}{=} r^2 \chi(\mathcal{O}_Y) + (r-1) c_1(E)^2 - 2r c_2(E)$$

$$\| $$

$$1 - \text{exp.dim.}\, \mathcal{M}_{[E]}$$

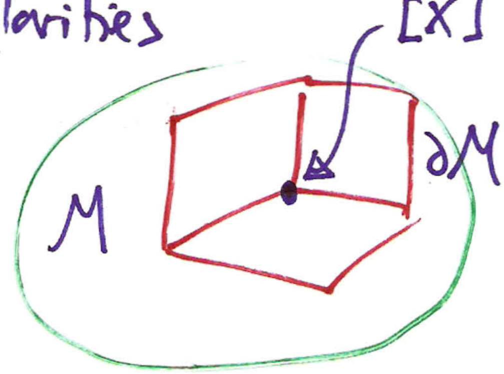
Can we use Donaldson/Seiberg-Witten to prove existence?

New surfaces of general type (Y. Lee, J. Park 2006)

$\pi_1(Y)=0$, $H^{2,0}(Y)=0$, Y general type.

$Y \rightsquigarrow X$ many Wahl singularities

$X \xleftarrow{\text{resolve}} \tilde{X}$ rational!



Use to construct new topological types of Y.

[Barlow 1984 $K_X^2 = 1$]

Lee, Park et al. $K_X^2 = 1,2,3,4$ (many examples)

$\boxed{Q4}$ Can we give complete classification?

Note: $Y \overset{\text{homeo}}{\simeq} Bl^n \mathbb{P}^2$, $n = 9 - K_Y^2$ (Freedman)

exp. dim $\mathcal{M} \overset{RR}{=} 2n-8$

"fake del Pezzo surfaces"