Speculations on Hodge theory

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Two areas where I see the potential for interesting developments are

- arithmetic aspects of Hodge theory, and
- singularities, i.e., degenerations of polarized Hodge structures.

The notations are those used in my talk [G2].

1 Arithmetic

Recall that a Mumford–Tate domain D_M is an open set in its compact dual $\check{D}_M$ — the latter is a rational homogeneous variety defined over $\mathbb{Q}$ (think of the upper half plane $\mathcal{H} \subset \mathbb{P}^1$). Thus a polarized Hodge structure (PHS) φ has an "upstairs" field of definition $k(\varphi)$. If $\mathcal{H}^{\bullet,\bullet} \subset T^{\bullet,\bullet}$ is a subalgebra then the Noether-Lefschetz locus

$$\mathrm{NL}_{\mathcal{H}} = \{\varphi \ : \ \mathrm{Hg}_{\varphi}^{\bullet,\bullet} \supseteq \mathcal{H}\}$$

is defined over $\mathbb{Q}$. Its components are defined over a number field and there is a Galois action on them. Let $\Gamma \subset M$ be an arithmetic subgroup and

$$\mathcal{M}_M := \Gamma \backslash D_M$$

be the moduli space of Γ-equivalence classes of PHS whose Mumford–Tate group is contained in M. In the classical case (Shimura varieties), $\mathcal{M}_M$ is projective and there is "arithmetic downstairs" related to arithmetic properties of automorphic forms. Let S be a quasi-projective variety defined over a number field k and

$$\Phi \colon S \to \mathcal{M}_M$$

a variation of Hodge structure whose monodromy group is $\mathbb{Q}$-Zariski dense in M. Then it is known that $\Phi^{-1}(\Gamma \backslash \mathrm{NL}_{\mathcal{H}})$ is an algebraic subvariety of S.

(∗) Is $\Phi^{-1}(\Gamma\backslash \mathrm{NL}_{\mathcal{H}})$ defined over k' with $[k' : k] < \infty$?

Example: If $\mathcal{X} \to S$ is a proper smooth family then the Hodge conjecture implies (∗). In the classical case, (∗) is a result of Deligne.

A cuspidal automorphic form f is an element in a cuspidal automorphic representation $U \subset L^2(M(\mathbb{Q})\backslash M(\mathbb{A}))$ that is a simultaneous eigenfunction for a finite-codimensional ideal in the center of the enveloping algebra $U(\mathfrak{m})$ and of a Hecke algebra.

(∗∗) Can one define the field of definition (a number field) of f? Does f have an algebro-geometric interpretation for $\Phi\colon S \to \mathcal{M}_M$?

In the classical case much is known. For a glimpse of what might be true in general see [C].

Note: f may be "realized" in automorphic cohomology $[f] \in H^d(\mathcal{M}_M, \mathcal{E})$, where in the non-classical case $d \neq 0, \dim \mathcal{M}_M$.

2 Singularities

For a Kato–Usui enlargement $D_M \subset D_{M,e}$ giving

$$\mathcal{M}_M \subset \mathcal{M}_{M,e} = \Gamma\backslash D_{M,e}$$

there is a very rich geometric (and possibly arithmetic) structure in the boundary components. Their cohomology, both classical (arising from a local system $\mathbb{V} \to \mathcal{M}_M$) and coherent (arising from the canonical extension of a homogeneous vector bundle $\mathcal{E}$), has been the objective of an extensive and rich theory (cf. [S] and [M] for recent works). Just as there are arithmetical consequences of the Hodge conjecture, there appear to also be topological aspects related to boundary cohomology of the "universal Néron model"

$$\begin{array}{ccc} \mathcal{J}_{M,e} & = & \hat{\Gamma}\backslash J_{M,e} \\ \downarrow & & \downarrow \\ \mathcal{M}_{M,e} & = & \Gamma\backslash D_{M,e} \end{array}$$

(cf. [Sa], [Sc], [Y]). Given a smooth projective variety X of dimension $2n$, a primitive Hodge class $\xi \in \mathrm{Hg}^\bullet(X)_{\mathrm{prim}}$, and an ample line bundle $L \to X$, we have

$$X \subset \mathbb{P}^{N_k} = \mathbb{P}H^0(\mathcal{O}_X(L^k))^\vee.$$

There is defined the singular locus

$$\operatorname{sing}\xi \subset (\mathbb{P}^{N_k})^\vee,$$

and (cf. [BFNP], [dCM])

$$\text{Hodge conjecture} \quad \iff \quad \operatorname{sing}\xi \neq \emptyset \text{ for } k \gg 0.$$

Regarding the meaning of $k \gg 0$, there is the question

$(***)$ Assuming the Hodge conjecture, is $k \geq (\text{constant})|\xi|^2$ an optimal bound?

This is true in the case $n = 1$.

The definition of $\operatorname{sing}\xi$ does not use the assumption that $\xi \in H^{2n}(X, \mathbb{Q})$ is *rational*; without this assumption one does not expect to have $\operatorname{sing}\xi \neq \emptyset$. The rationality assumption is used to construct a geometric object (normal function)

$$\nu_\xi \colon (\mathbb{P}^{N_k})^\vee \dashrightarrow \mathcal{J}_{M,e}.$$

Moreover, at least set-theoretically,

$$\operatorname{sing}\xi = \nu_\xi^{-1}(\Xi)$$

for a subvariety $\Xi \subset \mathcal{J}_{M,e}$. Thus, e.g., presumably (cf. [G])

$$(****) \qquad \nu_\xi^*([\Xi]) \neq 0 \quad \implies \quad \operatorname{sing}\xi \neq \emptyset.$$

If we let (X, ξ) vary over the Noether–Lefschetz locus of ξ in the moduli space of X, there is a possible converse to $(****)$ (loc. cit.).

In the classical case the cohomologies of objects over $\mathcal{M}_{M,e}$ (Shimura varieties in this case) are complex, and rich geometrically and arithmetically.

Only the very earliest glimpses of what might be the story in the non-classical case have been found. Even though some of the interesting specific questions are relatively easy to formulate, it is my sense that it may be difficult to attack them directly. Rather they might serve as a guide to investigate the general arithmetic and cohomological structure of the spaces $\Gamma\backslash D_{M,e}$ and of completed variations of Hodge structure $\Phi\colon \overline{S} \to \Gamma\backslash D_{M,e}$.

References

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