Mumford-Tate Groups and Mumford-Tate Domains

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This talk will discuss

- What are Mumford-Tate groups and Mumford-Tate domains?
- What are they good for?
- What is known about them?
- What would we like to know that isn't known, and why?

Outline

I. *Introductory material*

A. Polarized Hodge structures

B. Mumford-Tate groups

C. Mumford-Tate domains

II. *Hodge groups and Hodge domains*

A. Hodge representations

B. Hodge groups and Hodge domains

C. Classification

D. Examples

III. *Variation of Hodge structure and the structure theorem*

IV. *Some open issues*

I. Introductory material

A. *Polarized Hodge structures*

- V is a $\mathbb{Q}$-vector space
- $S^1 = \{z = e^{i\theta}\}$, $\bar{z} = z^{-1}$

Any representation

$$(*) \qquad \varphi : S^1 \to \mathrm{Aut}(V_{\mathbb{R}})$$

decomposes on $V_{\mathbb{C}}$ into eigenspaces

$$(**) \qquad \begin{cases} V_C = \oplus V^{p,q}, \ \overline{V}^{p,q} = V^{q,p} \\ \\ \varphi(z) = z^p \bar{z}^q = e^{i(p-q)} \ \text{ on } \ V^{p,q}. \end{cases}$$

Definition. A *Hodge structure of weight* n is given by $(*)$ where $p + q = n$ in $(**)$.

We set $C = \varphi(i)$ (*Weil operator*)

- $Q : V \otimes V \to \mathbb{Q}$, non-degenerate and ${}^tQ = (-1)^n Q$.

Definition. A *polarized Hodge structure* (V, Q, φ) *of weight* n is given by $\varphi : S^1 \to \mathrm{Aut}(V_{\mathbb{R}}, Q)$ as above and where the *Hodge-Riemann bilinear relations*

$$\begin{cases} \text{(i)} \quad Q(V^{p,q}, V^{p',q'}) = 0 \qquad p' \neq n-p \\ \text{(ii)} \quad Q(v, C\bar{v}) > 0, \qquad 0 \neq v \in V_{\mathbb{C}} \end{cases}$$

are satisfied.

Polarized Hodge structures admit the usual operations $(\oplus, \otimes, \mathrm{Hom})$ of linear algebra. They form a semi-simple abelian category.

Hodge's theorem: The primitive cohomology $H^n(X, \mathbb{Q})_{\text{prim}}$ of a smooth, complex projective algebraic variety has a (functorial) PHS of weight n.

B. *Mumford-Tate groups*

- $G = \mathrm{Aut}(V, Q)$ — this is a $\mathbb{Q}$-algebraic group — note that

$$\varphi : S^1 \to G(\mathbb{R}).$$

Definition I: The *Mumford-Tate group* M_φ associated to (V, Q, φ) is the smallest $\mathbb{Q}$-algebraic subgroup of G such that $\varphi : S^1 \to M(\mathbb{R})$.

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For a PHS (W, Q_W, φ_w) of even weight $n = 2m$, the *Hodge classes* are

$$\mathrm{Hg}_\psi = W \cap W^{m,m}.$$

Hodge conjecture: $\mathrm{Hg}(H^*(X, \mathbb{Q}))$ are represented by algebraic cycles

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Set

$$\begin{cases} T^{k,l} = V^{\otimes k} \otimes \check{V}^{\otimes l} \\ T^{\bullet,\bullet} = \oplus T^{k,l} \\ \mathrm{Hg}_{\varphi}^{\bullet,\bullet} = \text{Hodge tensors in } T^{\bullet,\bullet} \end{cases}$$

Definition II: The *Mumford-Tate group* M'_{φ} is the subgroup of $\mathrm{Aut}(V)$ that fixes all Hodge tensors.

Theorem: $M_{\varphi} = M'_{\varphi}$.

Example: $X = E_{\tau} = \mathbb{C}/\mathbb{Z} + \mathbb{Z}\tau$ and $V = H^1(E_{\tau}, \mathbb{Q})$. Then

$$M_{\tau} = \begin{cases} \mathbb{Q}(\tau)^* \text{ if } \tau \text{ is imaginary quadratic} \\ \mathrm{SL}_2 \text{ otherwise} \end{cases}$$

- M_φ is a reductive $\mathbb{Q}$-algebraic group. Thus it is an almost product

$$M = M_1 \times \cdots \times M_k \times T$$

where M_i is simple and T is a torus.

C. *Mumford-Tate domains*

Definition. Given (V, Q) and *Hodge numbers*

$$\mathfrak{h} = \{h^{p,q} = h^{q,p}, \sum_{p+q=n} h^{p,q} = \dim V\}$$

the *period domain*

$$D_{\mathfrak{h}} = \left\{ \begin{array}{c} \text{PHS's } (V, Q, \varphi) \\ \text{with } \dim V^{p,q} = h^{p,q} \end{array} \right\}.$$

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Given a reference point $\varphi \in D_{\mathfrak{h}}$,

$$D_{\mathfrak{h}} = G(\mathbb{R})/\widetilde{H}_\varphi$$

where $\widetilde{H}_\varphi = Z_G(\varphi(S^1))$ is the compact isotropy group fixing φ.

Definition. Given $\varphi \in D_{\mathfrak{h}}$ with MT-group M_φ, the *Mumford-Tate domain*

$$D_{M,\varphi} \subset D_{\mathfrak{h}}$$

is the $M_\varphi(\mathbb{R})$-oribit of φ.

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$$\rightsquigarrow D_{M,\varphi} = M_\varphi(\mathbb{R})/H_\varphi \text{ where}$$
$$H_\varphi = Z_{M_\varphi}(\varphi(S^1)).$$

Theorem: $D_{M,\varphi}$ is the component through φ of the subvariety

$$\mathrm{NL}_\varphi = \{\varphi' \in D_{\mathfrak{h}} : \mathrm{Hg}^{\bullet,\bullet}_{\varphi'} \supseteq \mathrm{Hg}^{\bullet,\bullet}_{\varphi}\}.$$

Classical case: $n = 1$ and $\Gamma \backslash D_{M,\varphi}$ are the (complex points of) *Shimura varieties*.

The *non-classical case* (or non-Shimura variety case) is comparatively in its early stage of development. For example, automorphic forms — especially their arithmetic aspects — is replaced by "automorphic cohomology", whose possible arithmetic and geometric meanings are a major open issue.

II. Hodge groups and Hodge domains

A. *Hodge representations*

- M is a a reductive, linear $\mathbb{Q}$-algebraic group

Definition. A *Hodge representation* (M, ρ, φ) is given by a $\mathbb{Q}$-vector space V and representation $/\mathbb{Q}$

$$\rho : M \to \mathrm{Aut}(V)$$

such that (i) there is an invariant bilinear form $Q : V \otimes V \to \mathbb{Q}$ with ${}^tQ = (-1)^n Q$, and (ii) there is

$$\varphi : S^1 \to M(\mathbb{R})$$

such that $(V, Q, \rho \circ \varphi)$ is a polarized Hodge structure of weight n.

Since M is an almost direct product of simple $\mathbb{Q}$-algebraic groups and an algebraic torus,[1] without essential loss of generality we shall assume that M is simple.

B. *Hodge groups and Hodge domains*

Definitions. (i) A *Hodge group* is an M that admits a Hodge representation.

(ii) A *Hodge domain* is $D_{M,\varphi} = M(\mathbb{R})/Z_\varphi$ where (M, ρ, φ) is a Hodge representation and $Z_\varphi = Z_M(\varphi(S^1)) \subset M(\mathbb{R})$.

[1] The torus case is basically the study of CM-Hodge structures; this is very interesting and will be briefly mentioned below.

Proposition: For any Hodge representation (M, ρ, φ), the corresponding Mumford-Tate domain is biholomorphic as a homogeneous complex manifold to the Hodge domain $D_{M,\varphi}$. Moreover, the infinitesimal period relations[2] correspond.

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Thus, we feel that Hodge groups and Hodge domains are the basic universal objects in Hodge theory.

C. *Classification*

Theorem: (i) M is a Hodge group if, and only if, there is $\varphi : S^1 \to M(\mathbb{R})$ such that

[2] To be defined later. Briefly, any variation of Hodge structure has tangent spaces contained in an invariant, canonical distribution $W \subset TD_{M,\varphi}$.

$(\mathfrak{m}, B, \operatorname{Ad}\varphi)$ is a polarized Hodge structure. (ii) If M is a Hodge group, then for any $\rho : M \to \operatorname{Aut}(V, Q)$, $(V, \pm Q, \rho \circ \varphi)$ is a polarized Hodge structure.

Theorem: The following are equivalent:

(i) M is a Hodge group

(ii) M is a semi-simple $\mathbb{Q}$-algebraic group such that $L^2(M(\mathbb{R}))$ has non-trivial discrete series summands.

(iii) For $\Gamma \subset M$ an arithmetic subgroup, $L^2(M(\mathbb{Q})\backslash M(\mathbb{A}))$ may have non-trivial cuspidal automorphic representation.

List of the real simple Lie algebras of Hodge groups:

A_r su(p, q), $p + q = r + 1$ and $0 \leqq p < q$
sl$_2$;

B_r so$(2p, 2q + 1)$, $p + q = r$;

C_r sp(p, q),* $p + q = r$ and $0 \leqq p, q \leqq r$
sp$(2r)$;

D_r so$(2p, 2q)$, $p + q = r$ and $0 \leqq p \leqq q \leqq r$
so$^*(2r)$; *

E_6 EII,* EIII; *

E_7 EV, * EVI, * EVII; *

E_8 EVIII, * EIX; *

F_4 FI, * FII; *

G_2 G. *

Those with $*$ were *not* known to be Lie algebras of Mumford-Tate groups.

It is harder to have a Hodge representation of odd weight. Following is a list of the simple Lie algebras for which there is such:

$$su(4k), so(4k+2) \quad \text{(compact cases)}$$
$$su(2p, 4k-2p), su(2p+1, 2q+1)$$
$$so(4p+2, 2q+1), so(2p, 2q)\text{for } p+q \text{ odd}$$
$$so^*(4k)$$
$$sp(n)$$
$$EV, EVII \quad (\text{for } E_7)$$

D. *Examples*

$\mathbf{G_2}$: For $M = G =$ non-compact real form of G_2, there are three distinct Hodge domains that give a diagram

$$\begin{array}{ccc} & D_{G,\varphi_3} & \\ \swarrow & & \searrow \\ D_{G,\varphi_1} & & D_{G,\varphi_2} \end{array}$$

In his famous 1905 paper, E. Cartan classified the 5-dimensional manifolds with a G_2-invariant exterior differential system. One is D_{G,φ_1} — the flat non-integrable 2-plane field[3] — and the other D_{G,φ_2} is a contact system. The above diagram is the *Cartan-Bryant incidence correspondence*: In the 4-plane contact field in

[3]The integrals of this system are one 2-sphere rolling on another without slipping and whose radii have the ratio 1:3.

D_{G,φ_2} there is a field F of rational normal cubics. Each point of D_{G,φ_1} corresponds to a $\mathbb{P}^1$ in D_{G,φ_2} that is tangent to F, and conversely. Variations of Hodge structure mapping to D_{G,φ_1} have a "dual" variation of Hodge structure mapping to D_{G,φ_2}, and vice-versa.[4] Such a pleasing picture certainly should have algebro-geometric significance.

[4]The dual variations of Hodge structure should be contact curves whose tangents are in F.

Mumford-Tate groups of Hodge structures of mirror quintic type

For any Hodge structure (V, φ) we set

$$\mathcal{E}_\varphi = \mathrm{End}(V, \varphi) = \left\{ \begin{array}{c} g : V \to V, \\ [g, \varphi] = 0 \end{array} \right\}$$ [5]

This is an algebra over $\mathbb{Q}$; if the Hodge structure is *simple* it is a division algebra. These are classified. Of particular interest are those when $\mathcal{E}_\varphi$ is a *CM-field* k; i.e. k is a purely imaginary quadratic extension of totally real number field k_0.

Definition. (V, φ) is a *CM Hodge structure* if $\mathcal{E}_\varphi$ is a CM-field with $[k : \mathbb{Q}] = \dim V$. In this case, $k^* \cong M_\varphi$ is an algebraic torus over $\mathbb{Q}$.

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[5] Note that $\mathcal{E}_\varphi$ preserves any polarization.

Definition. (V, Q, φ) is a *polarized Hodge structure of mirror quintic type* if the weight $n = 3$ and the Hodge numbers

$$h^{3,0} = h^{2,1} = 1.$$

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The period domain D has $\dim D = 4$ and the IPR is locally an *Engel system*

$$\begin{cases} dy - y'dx &= 0 \\ dy' - y''dx &= 0. \end{cases}$$

The Cattani-Kaplan-Schmid limiting mixed Hodge structures have been classified and a "Torelli theorem" for a mirror quintic has been proved.

Following is a table of all possible Mumford-Tate groups for simple such Hodge structures. Those with a "$*$" are non-classical.

type	$\dim \mathrm{NL}_\varphi$	$M_\varphi(\mathbb{R})^0$	$\mathcal{E}_\varphi$	$\mathrm{Gal}(k/\mathbb{Q})$
$(\mathrm{i})^*$	4	$\mathrm{Sp}(4)$	$\mathbb{Q}$	$\{e\}$
$(\mathrm{ii})^*$	2	$\mathrm{SL}_2 \times \mathrm{SL}_2$	$\mathbb{Q}(\sqrt{d})$	$\mathbb{Z}_2$
(iii)	1	$\mathcal{U}(1) \times \mathrm{SL}_2$	$Q(\sqrt{-d})$	$\mathbb{Z}_2$
$(\mathrm{iv})^*$	1	$\mathcal{U}(1) \times \mathrm{SL}_2$	$\mathbb{Q}(\sqrt{-d})$	$\mathbb{Z}_2$
(v)	1	SL_2	$\mathbb{Q}$	$\{e\}$
CM { (vi)	0	$\mathcal{U}(1) \times \mathcal{U}(1)$	k	$\mathbb{Z}_4$
(vii)	0	$\mathcal{U}(1) \times \mathcal{U}(1)$	k	$\mathbb{Z}_2 \times \mathbb{Z}_2$

In (v) the algebra of Hodge tensors is generated in degrees 2, 4; in all other cases they are generated in degree 2.

In general, one should picture a period domain as having a very rich configuration of arithmetically defined MT-sub-domains (e.g., CM polarized Hodge structures are dense).

III. *Variation of Hodge structure and the structure theorem*

Let $D_{\mathfrak{h}} = G(\mathbb{R})/H$ be a period domain and $\Gamma \subset G_{\mathbb{Z}} = G \cap (\operatorname{Aut} V_{\mathbb{Z}})$ where $V_{\mathbb{Z}} \subset V$ is a lattice.

Definition. A *variation of Hodge structure* is given by

$$\Phi : S \to \Gamma \backslash D_{\mathfrak{h}}$$

where S is a quasi-projective smooth algebraic variety and Φ is a locally liftable holomorphic mapping that satisfies the infinitesimal period relation (IPR)

$$\Phi_*(TS) \subset W \subset T(\Gamma \backslash D_{\mathfrak{h}}).$$

We think of $\Phi(s)$ as giving

$$V_{\mathbb{C}} = \oplus V_s^{p,q},$$

defined up to the action of Γ. Then

(i) $F_s^P = \bigoplus_{p' \geq p} V_s^{p',q'}$ varies holomorphically with $s \in S$.

(ii) $\frac{dF_s^P}{ds} \subseteq F_s^{p-1}$ gives W.

A family $\mathcal{X} \xrightarrow{\pi} S$ of smooth projective varieties $X_s = \pi^{-1}(s)$ gives a variation of Hodge structure where

$$\begin{cases} V = H^n(X_{s_o}, \mathbb{Q})_{\text{prim}} \\ \Gamma = \text{monodromy group} \\ F_s^p = F^p H^n(X_s, \mathbb{C}). \end{cases}$$

Definition. The *Mumford-Tate group* M_Φ *of the variation of Hodge structure* is defined to be $M_{\Phi(\eta)}$ where $\eta \in S$ is a generic point. Then

$$\Gamma \subset M_{\Phi(\eta)}$$

and we have

$$\begin{array}{l} M_\Phi = M_1 \times \cdots \times M_k \times M' \\ \quad \cup \\ \quad \Gamma = \Gamma_1 \times \cdots \times \Gamma_k. \end{array}$$

Structure theorem: We have

$$\Phi : S \to \Gamma_1 \backslash D_{M_1} \times \cdots \times \Gamma_k \backslash D_{M_k} \times D_{M'}$$

where the $D_{M'}$ factor is constant and where

$$\Gamma_i(\mathbb{Q}) = M_i.$$

If $\Gamma_{i,\mathbb{Z}} \subset M_i$ is the arithmetic group arising from $V_{\mathbb{Z}}$, then

- $\Gamma_i \subseteq \Gamma_{i,\mathbb{Z}}$
- the tensor invariants of Γ_i and $\Gamma_{i,\mathbb{Z}}$ are the same.

Question: Is Γ_i of finite index in $\Gamma_{i,\mathbb{Z}}$; i.e., *is* Γ *an arithmetic group?*

IV. **Some open issues**

Because the natural setting for variations of Hodge structure would seem to be Hodge domains, without reference to a particular Hodge representation, a natural issue is

Extend the Cattani-Kaplan-Schmid theory of degenerating Hodge structures, and the related Kato-Usui theory of enlargements of

$$\Phi : S \to \Gamma\backslash D\text{'s},$$

to the setting of Hodge domains.

The point is that since a Hodge domain will be realized in many different ways as Mumford-Tate sub-domains of various period domains, the C-K-S and K-U theories will need to be recast in purely algebraic group-theoretic terms.[6] This could lead to some simplification and isolation of the essential points of the above theories.

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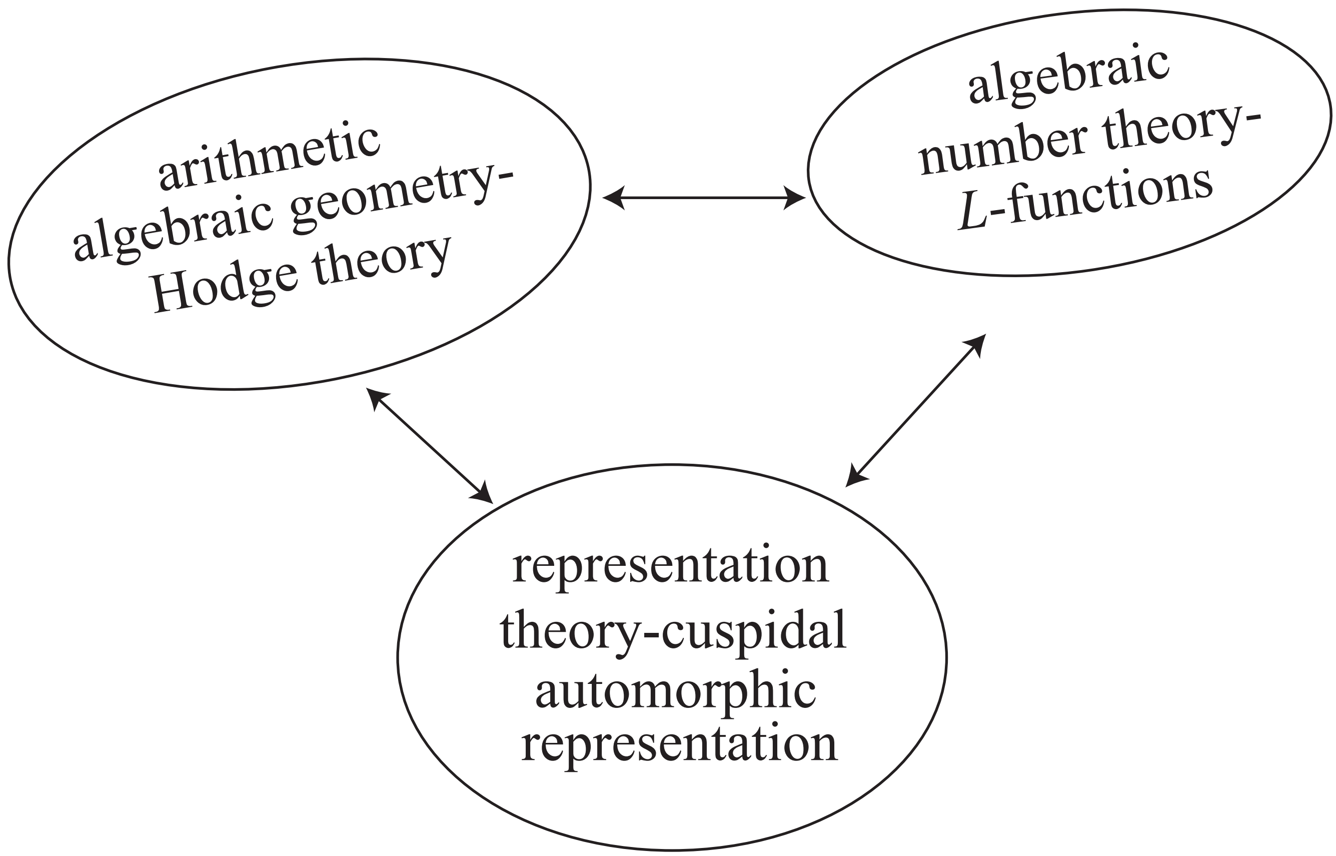
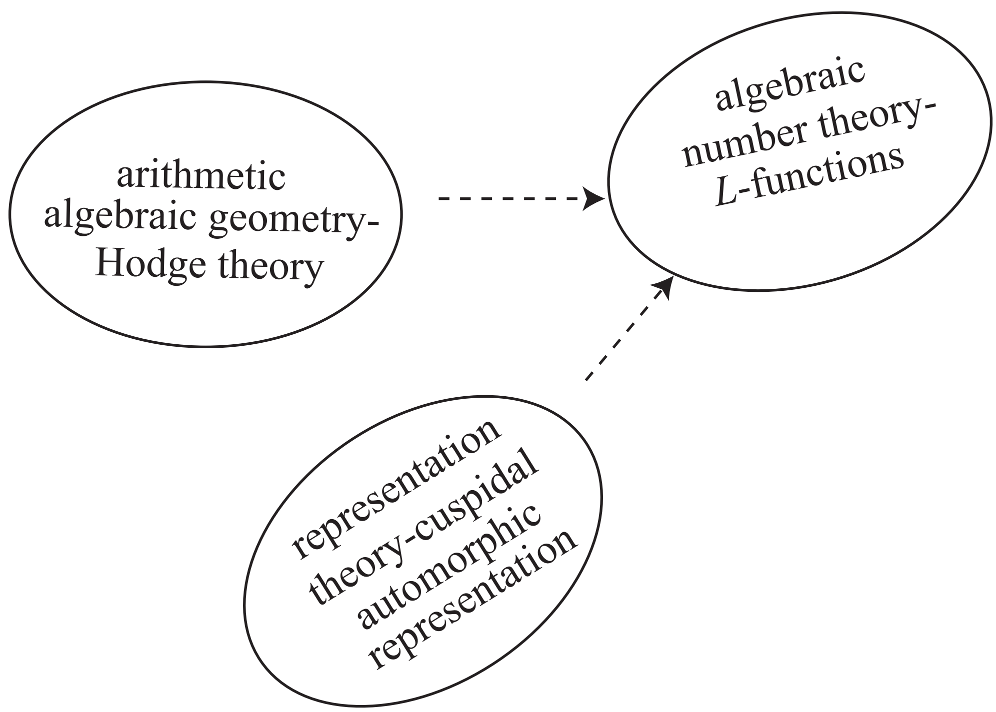
[6] For example, the index of unipotency of a local monodromy transformation will be bounded by the minimal weight for a Hodge representation of M.

In the classical case of Shimura varieties (the weight $n = 1$ case) there is an extensive and rich theory connecting

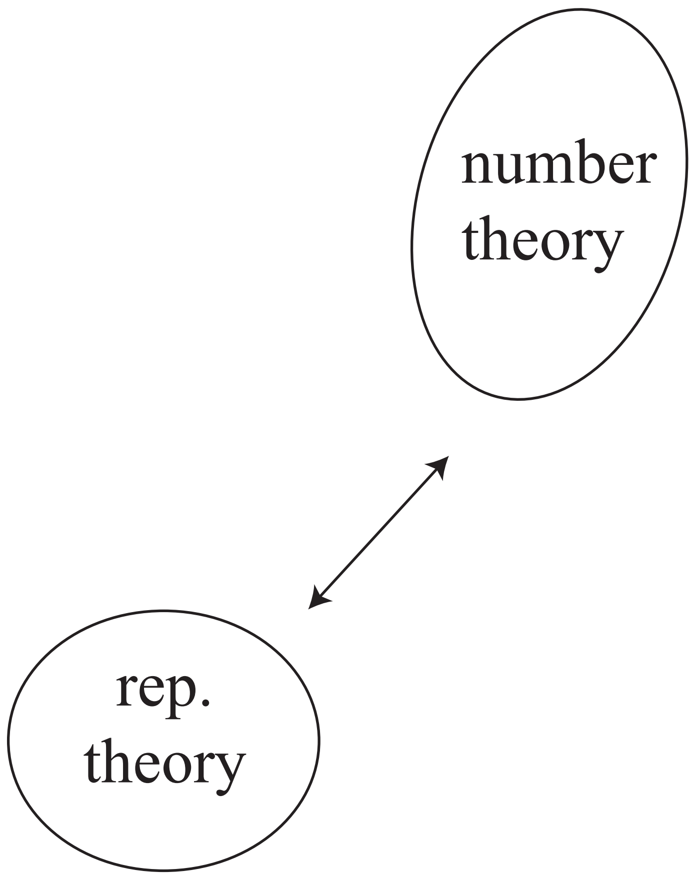


In the non-classical case, all three of these boxes are present but, e.g. in the case of cuspidal automorphic representations, are less highly developed. Almost entirely missing are the connections — the picture is roughly[7]

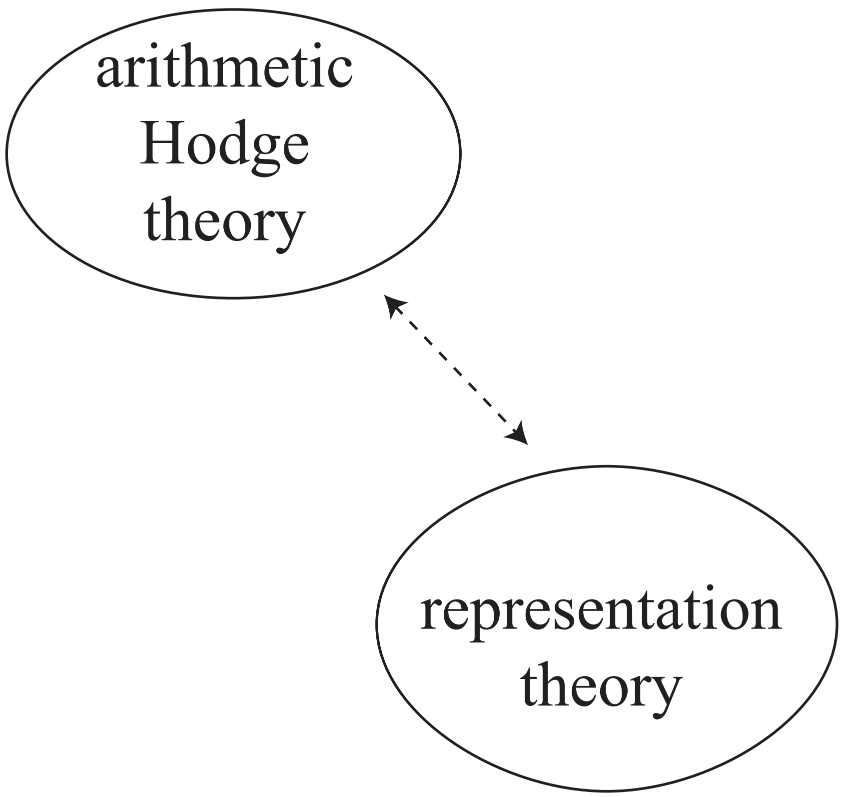


[7] The top dotted arrow assigns to a global VHS over $\overline{\mathbb{Q}}$ L-functions arising from the associated ℓ-adic Galois representations. The bottom dotted arrow associates an L-function to a cuspidal representation. The absence of a solid top arrow means that in general the analytic continuation and functional equation are missing.

It is the connections between these boxes — especially those coming from algebraic geometry/Hodge theory that enable the classical deep story



Since Hodge groups are exactly those for which one expects a theory of cuspidal automorphic representations — an implausibly accidental phenomenon — what is perhaps suggested is an effort to study the arithmetic aspects of global variations of Hodge structure by connecting



A good place to start might be the mirror quintic example above, or perhaps the non-classical Hodge domain when $M = \mathrm{SU}(2,1)$.

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