Two Questions about Rationally Connected Varieties

by

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1 Rationally Connected Varieties (Over $\mathbb{C}$)

Definition 1. A variety X is called *rationally connected* if two general points are connected by a rational curve.

When X is smooth and projective, this is equivalent to:

1. Two general points are connected by a chain of rational curves.
2. there is a map $f : \mathbb{P}^1 \to X$ such that $f^*T_X \cong \oplus\mathcal{O}(a_i), a_i \geq 1$.
3. For a finite number of points $p_1, \ldots, p_n \in X$, there is a map $f : \mathbb{P}^1 \to X$ such that $f^*T_X \cong \oplus\mathcal{O}(a_i), a_i \geq 1$ and $p_j \in f(\mathbb{P}^1)$.

- Examples include: (uni-)rational varieties, Fano varieties, etc.
- They are the higher dimensional analogues of rational surfaces.

2 Weak Approximation Conjecture

2.1 Background

Let $\pi : \mathcal{X} \to B$ be a flat surjective morphism from a smooth projective variety to a smooth projective curve such that a general fiber is rationally connected.

Theorem 2 (Graber-Harris-Starr)**.** *There is a section of $\pi : \mathcal{X} \to B$.*

- Equivalently, the generic fiber of $\mathcal{X} \to B$ has a rational point over the function field of the base curve.
- A general philosophy: a smooth rationally connected variety defined over a non-algebraically closed field should have many rational points, at least when the field is "good".

Conjecture 3 (Hassett-Tschinkel)**.** *The morphism π satisfies* weak approximation. *That is, for every finite sequence $(b_1, \ldots, b_m)$ of distinct closed points of B, for every sequence $(\widehat{s}_1, \ldots, \widehat{s}_m)$ of formal power series sections of π over b_i, and for every positive integer N, there exists a regular section s of π which is congruent to $\widehat{s}_i$ modulo $\mathfrak{m}^N_{B,b_i}$ for every $i = 1, \ldots, m$.*

2.2 What is known

- Places of good reductions
- Homogeneous spaces
- Stably rational varieties
- Low degree complete intersections
- Del Pezzo surface of degree at least 4
- Del Pezzo surface of degree 1 to 3 which are general

And many others.

A natural question is to ask if there are obstructions to weak approximation. But it is very difficult to calculate if some "expected obstructions" vanishes. So it is desirable to have some simple cases to test this.

2.3 A testing case

The construction:

- X, a smooth projective rationally connected variety
- $C \cong \mathbb{P}^1$, a smooth proper curve.
- G, a cyclic group of order l and acts on both X and C.
- $\mathcal{X} = (X \times C)/G$.
- $B = C/G \cong \mathbb{P}$

$$\begin{array}{ccc} X \times C & \xrightarrow{q} & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle\pi} \\ C & \rightarrow & B \end{array}$$

Theorem 4 (Tian)**.** *Weak approximation holds for the isotrivial family $\pi : \mathcal{X} \to B$ constructed above if and only if for every pair of fixed points (x, y) of X, there exists a G-equivariant map $f : \mathbb{P}^1 \to X$ such that $f(0) = x$ and $f(\infty) = y$.*

- For a general isotrivial family, there is a sufficient condition: existence of equivariant maps with respect to suitable subgroup of automorphism group.
- If weak approximation is true, there has to be equivariant curves. Conversely, existence of these curves proves weak approximation for isotrivial families!

Theorem 5 (Tian)**.** *Let X be a smooth degree d hypersurface in $\mathbb{P}^n$ $(n \geq d)$. Let G be a cyclic group of order l and act on X. Then for every pair of fixed points in X, there exist equivariant maps from $\mathbb{P}^1$ to X connecting them. In particular, weak approximation is satisfied for isotrivial families of hypersurfaces.*

- The proof suggests that when the variety varies in some moduli, this condition is likely to be satisfied.
- One could try to analyze equivariant curves in rigid Fano varieties with a group action.

3 Can We See Rationally Connectedness in Gromov-Witten Theory?

3.1 Algebraic geometry V.S. Symplectic geometry

- Every smooth projective variety is a symplectic manifold.
- The word "symplectic" is introduced by Weyl since it has the same meaning as "complex" in Greek.
- Gromov's theory of J-holomorphic curves builds a connection between them.

Algebraic geometry	Symplectic geometry
Blow ups and blow downs. By weak factorization, every birational map factors through blow up and blow downs.	Symplectic blow up and blow downs. Symplectic birational equivalence given by cobordisms. Birational projective manifolds are symplectic birational.
Uniruledness: $\exists$ rational curve through a general point. This is invariant under birational maps.	$GW^{X,\beta}_{0,n}([pt], \ldots) \neq 0$. For smooth projective varieties, equivalent to uniruledness. It is a symplectic birational invariant.
Rationally connectedness. It is a birational invariant.	$GW^{X,\beta}_{0,n}([pt], [pt], \ldots) \neq 0$. Not known if rationally connected varieties have this property. Not known if it is a symplectic birational invariant.
Minimal model program: classify varieties up to birational equivalence.	Symplectic birational geometry? Can we classify symplectic manifolds up to symplectic birational equivalence?

3.2 Is rationally connectedness a symplectic (birational) invariant?

Conjecture 6 (Kollar)**.** *Let X and X' be two symplectic equivalent smooth projective varieties (i.e. there is a symplectic diffeomorphism between them). Then X is rationally connected if and only if X' is rationally connected.*

- $GW^{X,\beta}_{0,n}([pt], [pt], \ldots) \neq 0$ will imply this conjecture.
- A major advance in this direction is the following theorem, due to Voisin.

Theorem 7 (Voisin)**.** *Let X be a smooth 3-dimensional rationally connected variety. If X is Fano or $b_2(X) = 2$, then X' is also rationally connected.*

When X is a Fano threefold, we actually have the following stronger results:

Theorem 8 (Tian)**.** *If X is a Fano threefold, then there are nonzero GW-invariants of the form $GW^{X,\beta}_{0,n}([pt], [pt], \ldots)$.*

- Fano threefolds have been completely classified. But the proof of the above theorem does *NOT* use the classification. Instead, the proof only uses results in birational geometry.