Fano Hypersurfaces in Positive Characteristics

by

Yi Zhu

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1 Introduction and Background

A basic question in birational geometry is to determine if a variety is birational to a projective space. Such varieties are called rational varieties. However, this question has only complete answers in curves and surfaces since we have the classification. In higher dimension, we do not have a good criterion if a variety is rational or not. There is another generalization of rational surfaces to higher dimensions, called rationally connected varieties. They are much easier to deal with and contain lots of interesting varieties, e.g. Fano varieties (Fano means anticanonical bundle ample).

Let X be a smooth proper variety over an algebraically closed field k of arbitrary characteristic.

Definition 1. X is called *rationally chain connected* if for a general pair of points there exists a chain of rational curves passing through them.

X is called *rationally connected* if for a general pair of points there exists a rational curve passing through them.

X is called *separably rationally connected* if there exists a very free rational curve $f : \mathbb{P}^1 \to X$, i.e., the pullback of the tangent bundle of X via f is ample over $\mathbb{P}^1$.

Proposition 2. *Separably rationally connected implies rationally connected. Rationally connected implies rationally chain connected. In characteristic 0, the above three definitions are equivalent.*

Example 3. Rational varieties are rationally connected. Rationally connected surfaces are rational surfaces.

Theorem 4 (Campana, Kollár-Miyaoka-Mori)**.** *Every smooth Fano variety is rationally chain connected. In characteristic 0, Fano varieties are (separably) rationally connected.*

Example 5. Smooth Fano hypersurfaces, i.e., of degree no more than n in $\mathbb{P}^n$, are (separably) rationally connected in characteristic 0.

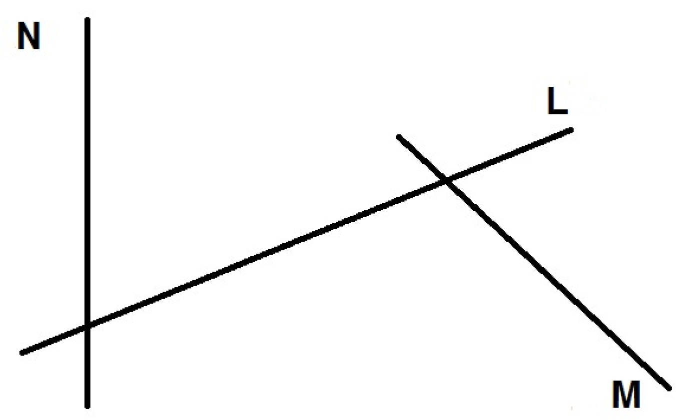
Theorem 6 (__)**.** *Let X be a general Fano hypersurface in $\mathbb{P}^n$ over an algebraically closed field of characteristic p. Then X is (separably) rationally connected.*

Question 7. *Is every smooth Fano hypersurface (separably) rationally connected in positive characteristic?*

Remark 8. The above question is false for general Fano varieties. To prove (separably) rationally connectedness, the standard argument is to use the existence of lots of free curves. In positive characteristic, there are Fano varieties which are not separably uniruled, due to Kollár.

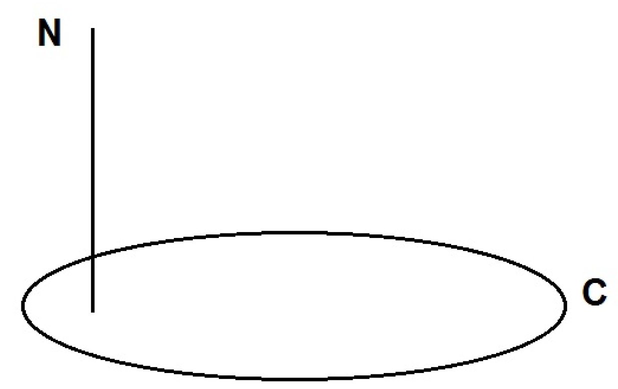
2 A Simple Example

We want to illustrate the cubic surface example which gives some hint of the proof. To prove separably rationally connectedness of a variety, it suffices to show that we can find a very free curve on it. Let us consider a smooth cubic surface in $\mathbb{P}^3$. Every smooth cubic surface contains 27 lines. Since we know the exact configuration of the 27 lines, we can choose three lines on it, L, M, and N such that M and N intersect L at two distinct points and they do not lie in the same plane.



Fix a configuration of 3 lines

For each line, the normal bundle is negative. However, if we put two lines together L and M, they are a specialization of a pencil of conics in the cubic surface. By adjunction formula, the normal bundle increase by degree one, which is trivial. For a general conic C in this pencil, it intersects N at one point. Finally, Take the union of the conic and the line N. If we take a general element of the corresponding linear system, we will get a rational normal curve of degree 3 lying in the cubic with normal bundle ample. Thus we get a very free curve.



After deforming $L \cup M$

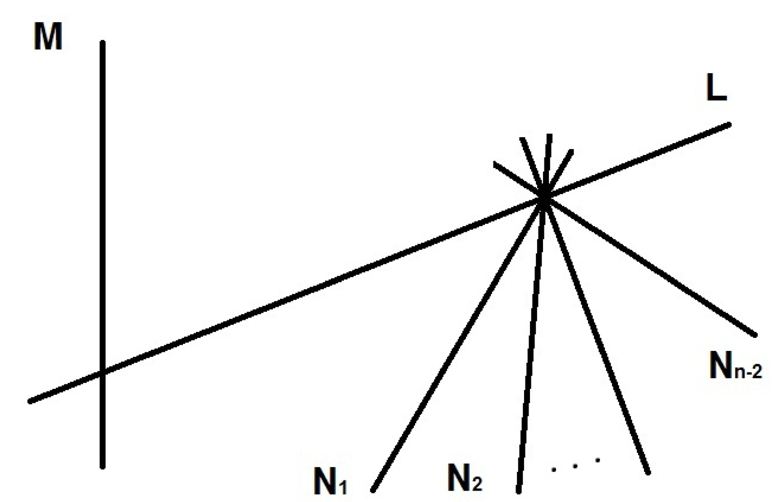
From this example, to find a rational curve with ample normal bundle, we can start with a line and glue some lines on it and deform the singular one to a smooth rational curve. Although the normal bundle of each component is negative, we can still add the positivity by increasing the degree, i.e., the number of components.

3 Sketch of the Proof

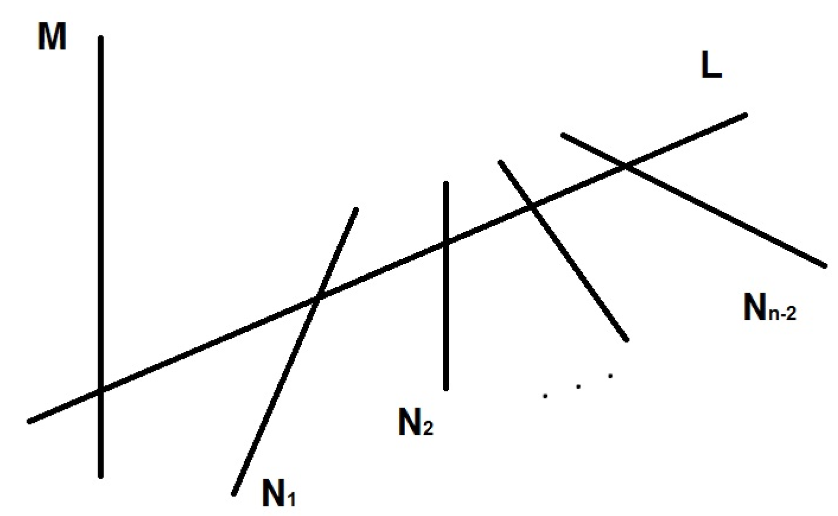
For the higher dimensional case, two problems arise. First, we can not use the linear system to deform the curves. However, we can use the curve deformation argument instead, which is developed by Kollár-Miyaoka-Mori and Graber-Harris-Starr. Second, since the normal bundle of a rational curve is a vector bundle, to add positivity, one needs to choose the gluing tails more carefully.

Since separably rationally connectedness is an open condition in families of varieties, the theorem follows if we can find a specific hypersurface having a very free curve. In the following, we will only consider hypersurfaces of degree n in $\mathbb{P}^n$. The remaining cases are easily from inductions by taking hyperplane sections.

Step 1 Find a hypersurface containing the following singular curve in its smooth locus such that the behavior of normal bundles of the lines are good.



Step 2 Deform the hypersurface and the curve into a hypersurface containing a comb with tails in the general directions of the normal bundle of the handle.



Step 3 Deform the comb to get a very free curves by the standard deformation argument of Graber-Harris-Starr.