On the maximal rank conjecture for line bundles of extremal degree

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1 Introduction

1.1 Background

A central problem in curve theory is to describe algebraic curves in a given projective space $\mathbb{P}^r$ ($r \geq 3$) with fixed genus and degree. For instance, one wants to describe the ideal of a curve $C \subset \mathbb{P}^r$, and in particular, to know the Hilbert function of C, or in geometric terms, how many independent hypersurfaces of each degree C lies on. A major open problem here is the maximal rank conjecture:

Conjecture 1. *(**Maximal Rank Conjecture**) For fixed d, g, $r \geq 3$, let C be a general curve of genus g and $|L|$ be a general g^r_d on C, then the multiplication map*

$$\mathrm{Sym}^k H^0(C, L) \xrightarrow{\mu^k} H^0(C, L^k) \tag{1.1}$$

is of maximal rank (either injective or surjective) for any $k \geq 1$.

In the case $|L|$ gives an embedding of C into $\mathbb{P}^r$, $\mathrm{Sym}^k H^0(C, L)$ is the space of homogeneous polynomials of degree k in $\mathbb{P}^r$ and $\mathrm{Ker}(\mu^k)$ is just the subspace consisting of those vanishing on C. Since on a general curve C, L^k is always non-special for $k \geq 2$, the dimension of the domain and target of μ^k are constants only depending on k, d, r and g. Therefore, the maximal rank conjecture (MRC) simply says that the number of independent hypersurfaces containing C is as small as it could be.

1.2 What is known

So far, only little is known about this conjecture. Ballico and Ellia proved the conjecture for non-special linear series. So the main interest now is in the case of special linear series on general curves.

In another development, Green and Lazarsfeld proved that any very ample line bundle L with

$$deg\ L \geq 2g + 1 - 2h^1(L) - Cliff(C)$$

on C is projectively normal, where $Cliff(C)$ is the clifford index of C:

$$Cliff(C) := \min\{\ deg(A) - 2r(A) \mid A \text{ line bundle on } C,\ h^0(A) \geq 2,\ h^1(A) \geq 2\}.$$

It is also showed by Green and Lazarsfeld that the bound $2g + 1 - 2h^1(L) - Cliff(C)$ is the best possible. There are line bundles of degree one less than this bound which are not normally generated. We say a line bundle L on C has **extremal degree** if

$$deg\ L = 2g - 2h^1(L) - Cliff(C).$$

that is

$$Cliff(L) = Cliff(C).$$

2 Main results

On the other hand, if the maximal rank conjecture were true, we should still expect projective normality for general line bundles of extremal degree on general curves. Thus the extremal degree range should be thought of as the first case to test the maximal rank conjecture.

For line bundles of extremal degree on general curves. There are four cases according to the value of $h^1(L)$:

(1) $h^1(L) = 0$. L is non special and this case follows from a result of Ballico and Ellia.

(2) $h^1(L) = 1$. If $g = 2l$ even, L is a g^l_{3l-1}; if $g = 2l + 1$ odd, L is a g^l_{3l}.

(3) $h^1(L) = 2$. If $g = 2l$ even, L is a g^{l-1}_{3l-3}; if $g = 2l + 1$ odd, L is a g^{l-1}_{3l-2}.

(4) $h^1(L) \geq 3$. The Brill-Noether number is negative. There are no such g^r_d's $(r \geq 3)$ on a general curve.

We prove the maximal rank conjecture for the remaining open case (2) and (3).

Theorem 2. *(Wang) Let C be a general curve of genus g ($g \geq 10$ if g even, $g \geq 13$ if g odd), L be a general line bundle of extremal degree on C, then (C, L) satisfies the MRC.*

3 The idea of proof

We apply a new method, using deformation theory, to prove theorem 2. There are several steps.

3.1 Reduction

In our degree and genus range, the MRC for L is equivalent to the statement that L is projectively normal. By a theorem of Green, in our degree range, $H^0(L) \otimes H^0(L^k) \to H^0(L^{k+1})$ is surjective for any $k \geq 2$. Thus to show such L is projectively normal, it suffices to show the multiplication map μ^2 in (1.1) is surjective.

3.2 Deformation theory

Suppose we have a one parameter family of pairs $(C_t, L_t) \in \mathcal{W}^r_d$, specializing to some (C_0, L_0) (C_0 could be singular) with $\mu^2(0)$ not necessarily of maximal rank. This means the dimension of $\mathrm{Ker}(\mu^2(0))$ is bigger than expected. We would like to knock down this dimension by showing that only a expected number of independent sections of $\mathrm{Ker}(\mu^k(0))$ can extend to $\mathrm{Ker}(\mu^k(t))$. To this end, we first set up some machinery which measures the obstructions for elements of $\mathrm{Ker}(\mu^k(0))$ to extend to $\mathrm{Ker}(\mu^k(t))$. We constructed obstruction maps

$$\delta_1 : \mathrm{Ker}(\mu^2(0)) \longrightarrow \mathrm{Coker}(\mu^2(0))$$

and inductively

$$\delta_{n+1} : \mathrm{Ker}(\delta_n) \longrightarrow \mathrm{Coker}(\delta_n)$$

such that an element $s \in \mathrm{Ker}(\mu^2(0))$ extends to $\mathrm{Ker}(\mu^2(t))$ modulo t^{n+1} if and only if $\delta_i(s) = 0$ for $i = 0, ..., n$.

For the decreasing sequence

$$\mathrm{Ker}(\mu^2(0)) \supset \mathrm{Ker}(\delta_1) \supset ... \supset \mathrm{Ker}(\delta_n) \supset ...$$

if we can show that the vector space $V = \bigcap_i \mathrm{Ker}(\delta_i)$ consisting of elements which deform to $\mathrm{Ker}(\mu^2(t))$ to any order is of "correct dimension", then $\mu^2(t)$ is of maximal rank. Said differently, it suffices to prove that δ_n is of maximal rank for some $n \in \mathbb{Z}_+$.

3.3 Computation of obstructions

We find a nice singular curve C_0 on which the computation of the obstruction maps is surprisingly simple. The information in the obstruction maps δ_n are captured by the natural multiplication map

$$\kappa_n : H^0(C_0, L_n) \otimes H^0(C_0, L_{-n}) \longrightarrow H^0(C_0, L_0^2)$$

in the following theorem:

Theorem 3. *(Wang) Let $\mathcal{L} \to \mathcal{C}$ be the total space of a one parameter family $(C_t, L_t) \in \mathcal{W}^r_d$ degenerating to (C_0, L_0) with $C_0 = X \cup Y$ a nodal curve consisting of two smooth curves of genus g_X, g_Y meeting at a point p and $\mathcal{C}$ is smooth. Write $L_n = \mathcal{L}(nY)|_{C_0}$. Suppose all (global) sections of L_n extend to L_t for $n = 0, ..., a$ and the natural map*

$$\oplus^a_{n=0} H^0(C_0, L_n) \otimes H^0(C_0, L_{-n}) \xrightarrow{\oplus \kappa_n} H^0(C_0, L_0^2) \tag{3.2}$$

is surjective (resp. of rank $= dim_{\mathbb{C}} \mathrm{Sym}^2 H^0(L_0)$) for some $a \in \mathbb{Z}_+$, then the multiplication map $\mu^2(t)$ is surjective (resp. injective) for small $t \neq 0$.

Notice that κ only depends on (C_0, L_0), not on the actual family specializing to it. It seems to the author that such a simple way to describe higher order obstructions is new and should have a lot more applications.

The significance of theorem 3 is that we are now reduced to finding a smoothable (C_0, L_0) such that all sections of L_n extend to the nearby fiber and $\oplus^a_{n=0}\kappa_n$ (instead of $\kappa_0 = \mu^2(0)$) is of required rank. By making a good choice of (C_0, L_0), we manage to prove theorem 2 by showing that κ in theorem 3 is surjective.

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