A Compactification of the Space of Algebraic Maps from $\mathbf{P}^1$ to a Grassmannian

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1 Motivation and Background

Let V be a vector space of dimension n, $\mathrm{Gr} := \mathrm{Gr}(k, V)$ the Grassmannian, and $M_d := \mathrm{Mor}_d(\mathbf{P}^1, \mathrm{Gr})$ the moduli space of degree d morphisms $f : \mathbf{P}^1 \to \mathrm{Gr}$ for $d \geq 0$. The spaces M_d are of interest not only in algebraic geometry but also in theoretical physics and in control theory. M_d is a smooth non-compact quasi-projective variety for $d \geq 1$, and is usually studied through its compactifications.

If K is a compactification of M_d, then we wish that K is as "nice" as M_d and the boundary $K \setminus M_d$ is as simple as possible. This idea motivates the search for a compactificaition K satisfying the following properties:

1. K is a smooth projective variety.
2. The boundary $K \setminus M_d$ is a divisor with normal crossings.
3. K has a moduli space structure compatible with that on M_d.

Similar examples of such compactifications are the spaces of complete quadrics, complete collineations, complete correlations, etc., which compactify the spaces of non-degenerate quadrics, non-degenerate collineations, non-degenerate correlations, etc., respectively.

For M_d, such a compactification is not known. Among the various compactifications of M_d, two types are prominent. One is given by the Grothendieck Quot scheme, $Q_d := \mathrm{Quot}^{n-k,d}_{V_{\mathbf{P}^1}/\mathbf{P}^1}$, which is a fine moduli space parametrizing quotient sheaves of $V_{\mathbf{P}^1} := V \otimes \mathcal{O}_{\mathbf{P}^1}$ of rank $n-k$ and degree d. It satisfies Properties 1 and 3 but not 2. The other type is given by the moduli spaces of stable maps, e.g., $\overline{M}_{0,3}(\mathrm{Gr}, d)$ and $\overline{M}_{0,0}(\mathbf{P}^1 \times \mathrm{Gr}, (1, d))$. They satisfy Properties 2 and 3 but not 1.

We will provide an explicit construction of a compactification for M_d, and prove that it satisfies Properties 1 and 2. In the future work we will show it actually satisfies Property 3.

2 The Method of Construction

Our construction is obtained from the Quot-scheme compactification Q_d through a sequence of blowing-ups along "bad" loci in the boundary. We first review how the Quot scheme compactifies M_d. Let $V_{\mathrm{Gr}} \twoheadrightarrow Q$ be the universal quotient bundle on Gr. Then we have a 1-to-1 correspondence

$$M_d \overset{1-1}{\longleftrightarrow} \{\text{quotient bundles } V_{\mathbf{P}^1} \twoheadrightarrow W \text{ of rank } n-k \text{ and degree } d \text{ on } \mathbf{P}^1\}$$
$$(f : \mathbf{P}^1 \to \mathrm{Gr}) \mapsto (V_{\mathbf{P}^1} = f^*V_{\mathrm{Gr}} \twoheadrightarrow f^*Q)$$

By allowing W to have torsion, one obtains the compactification Q_d, which comes equipped with a universal exact sequence

$$0 \to \mathcal{E} \to V_{\mathbf{P}^1 \times Q_d} \to \mathcal{F} \to 0 \tag{2.1}$$

where $\mathcal{F}$ is flat over Q_d of rank $n-k$ and relative degree d.

Theorem 1 (Strømme). *Q_d is an irreducible smooth projective variety.*

Every point $q \in Q_d$ is represented by the quotient $V_{\mathbf{P}^1 \times \{q\}} \twoheadrightarrow \mathcal{F}_q$ where $\mathcal{F}_q := \mathcal{F}|_{\mathbf{P}^1 \times \{q\}}$.

The boundary $Q_d \setminus M_d$, which consists of points q such that $\mathcal{F}_q$ is not locally free, is not a divisor and has complicated singularities. The construction is essentially a desingularization of the boundary by blowing-ups. There is a natural filtration of the boundary by closed subschemes

$$Z_{d,0} \subset Z_{d,1} \subset \cdots \subset Z_{d,d-1} = Q_d \setminus M_d$$

Set-theorectically, $Z_{d,r}$ consists of points q such that $\mathcal{F}_q$ has a torsion of degree $\geq d-r$. We must give them appropriate scheme structures. First dualizing the exact sequence (2.1):

$$0 \to \mathcal{F}^\vee \to V^\vee_{\mathbf{P}^1 \times Q_d} \to \mathcal{E}^\vee \to \mathcal{E}xt^1(\mathcal{F}, \mathcal{O}) \to 0$$

where $\mathcal{O} := \mathcal{O}_{\mathbf{P}^1 \times Q_d}$. Then applying $\pi_*(- \otimes p^*\mathcal{O}_{\mathbf{P}^1}(m))$ to $V^\vee_{\mathbf{P}^1 \times Q_d} \to \mathcal{E}^\vee$ to obtain a homomorphism:

$$\rho_{d,m} : \pi_* V^\vee_{\mathbf{P}^1 \times Q_d}(m) \to \pi_* \mathcal{E}^\vee(m)$$

Then we define $Z_{d,r}$ to be the zero locus of $\bigwedge^{k(m+1)+r+1} \rho_{d,m}$ for any $m \gg 0$. It turns out that the locally closed subschemes

$$Z_{d,0},\ (Z_{d,1} \setminus Z_{d,0}),\ \ldots,\ (Z_{d,d-1} \setminus Z_{d,d-2}),\ M_d$$

form a flattening stratification of Q_d by the torsion sheaf $\mathcal{E}xt^1(\mathcal{F}, \mathcal{O})$. This fact will be useful in the proof.

Then we perform a sequence of blowing-ups as follows:

$$\begin{array}{ccccccccc}
Z^{d-1}_{d,0} & & Z^{d-1}_{d,1} & & \cdots & & Z^{d-1}_{d,d-1} & \subset & Q^{d-1}_d \\
\downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \ \mathrm{Bl}_{Z^{d-2}_{d,d-1}} \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \ \mathrm{Bl}_{Z^0_{d,1}} \\
Z^0_{d,0} & & Z^0_{d,1} & \subset & \cdots & \subset & Z^0_{d,d-1} & \subset & Q^0_d \\
\downarrow & & \downarrow & & \cdots & & \downarrow & & \downarrow \ \mathrm{Bl}_{Z_{d,0}} \\
Z_{d,0} & \subset & Z_{d,1} & \subset & \cdots & \subset & Z_{d,d-1} & \subset & Q_d
\end{array}$$

Our main theorem reads:

Theorem 2. (i) *Q^{d-1}_d is a smooth projective variety and contains M_d as a dense open subset.*
(ii) *$Q^{d-1}_d \setminus M_d = Z^{d-1}_{d,0} \cup \cdots \cup Z^{d-1}_{d,d-1}$, and $\sum_r Z^{d-r}_{d,r}$ is a divisor with normal crossings.*

3 The Special Case of $k = 1$ as an Example

When $k = 1$, the Grassmannian $\mathrm{Gr}(k, V)$ is a projective space $\mathbf{P}^{n-1}$. In this case, the Quot scheme together with the construction has a simple description. This case has been carried out in a paper joint with Y. Hu and J. Lin for publication.

When $k = 1$, $M_d = \mathrm{Mor}_d(\mathbf{P}^1, \mathbf{P}^{n-1})$. Each degree d morphism $f : \mathbf{P}^1 \to \mathbf{P}^{n-1}$ is given by a n-tuple $(f_1, \ldots, f_n)$ of degree d homogenous polynomials in x, y with no common factors up to a nonzero scalar multiple:

$$f([x : y]) = [f_1(x, y) : \cdots : f_n(x, y)].$$

Suppose $f_i = \sum_{j=0}^d a_{ij} x^{d-j} y^j$. Then f corresponds to the point $[\cdots : a_{ij} : \cdots] \in \mathbf{P}^{n(d+1)-1}$. By allowing the f_i's to have common factors, we obtain a compactification $\mathbf{P}^{n(d+1)-1}$, which is exactly the Quot scheme Q_d with $k = 1$. The homomorphism $\rho_{d,m}$ is represented by the $(m+d+1) \times n(m+1)$ matrix

$$\left[\begin{array}{ccccccccccc}
a_{10} & \cdots & a_{n0} & & & & & & & & \\
a_{11} & \cdots & a_{n1} & a_{10} & \cdots & a_{n0} & & & & & \\
\vdots & \cdots & \vdots & a_{11} & \cdots & a_{n1} & \cdots & & & & \\
a_{1d} & \cdots & a_{nd} & \vdots & \cdots & \vdots & \cdots & \cdots & & & \\
 & & & a_{1d} & \cdots & a_{nd} & \cdots & \cdots & a_{10} & \cdots & a_{n0} \\
 & & & & & & \cdots & \cdots & a_{11} & \cdots & a_{n1} \\
 & & & & & & & \cdots & \vdots & \cdots & \vdots \\
 & & & & & & & & a_{1d} & \cdots & a_{nd}
\end{array}\right]$$

Hence $Z_{d,r}$ is defined by the homogeneous ideal generated by the $(m+r+2) \times (m+r+2)$ minors of the matrix for any $m \gg 0$. Set-theoretically,

$$Z_{d,r} = \{[\cdots : a_{ij} : \cdots] \mid f_1, \ldots, f_n \text{ have a common factor of degree } \geq d-r\}$$

By factoring out the common factors, we have a morphism

$$\varphi : \mathbf{P}^{d-r} \times Q_r \twoheadrightarrow Z_{d,r} \subset Q_d \quad \text{by} \quad \varphi([h], [g_1, \ldots, g_n]) = [hg_1, \ldots, hg_n]$$

whose restriction gives an isomorphism

$$\varphi : \mathbf{P}^{d-r} \times M_r \xrightarrow{\sim} Z_{d,r} \setminus Z_{d,r-1}$$

where $\mathbf{P}^{d-r}$ is the space common factors. It follows that $Z_{d,r}$ are irreducible, and $Z_{d,r} \setminus Z_{d,r-1}$ are smooth. In particular $Z_{d,0}$ is smooth, and hence Q^0_d is smooth. We show Q^1_d is smooth by showing that $Z^0_{d,1}$ is nonsingular, which follows from the isomorphism

$$\mathbf{P}^{d-1} \times Q^0_1 \xrightarrow{\sim} Z^0_{d,1} \subset Q^0_d$$

Keep going like this, we can actually arrive at a proof for this special case.

4 Sketch of the Proof

We have already seen a pattern of the proof in the special case $k = 1$. The general case has similar pattern. In the general case, $\mathbf{P}^{d-r} \times Q_r$ is replaced with a relative Quot scheme $Q_{d,r} := \mathrm{Quot}^{0,d-r}_{\mathcal{E}/\mathbf{P}^1 \times Q_r/Q_r}$ parametrizing quotient torsion sheaves of $\mathcal{E}$ on $\mathbf{P}^1 \times Q_r$ flat over Q_r of relative degree $d-r$. As expected, $Q_{d,r}$ is a smooth projective variety. Similarly, we have a morphism and an isomorphism

$$Q_{d,r} \twoheadrightarrow Z_{d,r} \hookrightarrow Q_d, \quad Q_{d,r} \times_{Q_r} M_r \xrightarrow{\sim} Z_{d,r} \setminus Z_{d,r-1}$$

whose proof relies on the flattening stratification property of $Z_{d,r} \setminus Z_{d,r-1}$. Then similarly we have the isomorphisms

$$Q_{d,0} \cong Z_{d,0}, \quad Q_{d,1} \times_{Q_1} Q^0_1 \cong Z^0_{d,1}, \quad \cdots, \quad Q_{d,r} \times_{Q_r} Q^{r-1}_r \cong Z^{r-1}_{d,r}, \quad \cdots$$

By using induction on r, we can show that the proper transforms $Z^{r-1}_{d,r}$ as well as the blowups Q^r_d's are smooth. In particular, Q^{d-1}_d is smooth. The proof for the normal crossing also follows from the above isomorphisms.