Toric vector bundles and Okounkov bodies

Jose Gonzalez - University of Michigan

1 Toric varieties and toric vector bundles

A normal variety X is a *Toric Variety* if the torus $T \cong (k^*)^n$ is an open subset of X, and T acts on X extending the natural action of T on itself.

Key tool: $X \iff$ Certain collection of rational polyhedral cones.
Properties of $X \iff$ Properties of collection of cones.

For example: Projectiveness, completeness, smoothness, etc. are understood in combinatorial terms. $\mathrm{Cl}(X)$ and $\mathrm{Pic}(X)$ can be computed using only T-invariant divisors. To each T-invariant divisor D, we can associate a polytope $P_D \subseteq \mathbf{R}^n$ such that

$$H^0\big(X, \mathcal{O}_X(D)\big) = \bigoplus_{u \in P_D \cap \mathbf{Z}^n} k\chi^u.$$

A vector bundle $\pi : \mathcal{E} \to X$ over a TV X is a *Toric Vector Bundle* if the total space of $\mathcal{E}$ has an action of T compatible with π and linear on its fibers.

Key tool: Klyachko's Equivalence: TVB $\pi : \mathcal{E} \to X \iff$ Vector space E with a collection of filtrations: (D T-invariant divisor, $i \in \mathbf{Z}$)

$$\cdots E_D(i-1) \supseteq E_D(i) \supseteq E_D(i+1) \supseteq \cdots$$

Satisfying compatibility: $\forall$ fixed point $p \in X$, $\exists$ decomposition $E = \bigoplus_{u \in \mathbf{Z}^n} E_u$, s.t. if $p \in D$, the filtration $\{E_D(i)\}$ can be recovered as $E_D(i) = \sum_{\langle u, v_D \rangle \geq i} E_u$.

2 Questions

We will study the variety $\mathbf{P}(\mathcal{E})$, where $\mathcal{E}$ is a rank 2 TVB. Questions:

1. Can we describe the Okounkov bodies of all line bundles on $\mathbf{P}(\mathcal{E})$ with respect to some suitable flag of subvarieties?
2. Is $\mathbf{P}(\mathcal{E})$ a Mori Dream Space?

Answers: 1. Yes!! 2. Yes!!

3 Okounkov bodies [Okounkov; Lazarsfeld-Mustaţă]

Let X be a projective variety of dimension n.

- Fix a flag of smooth subvarieties $Y_\bullet$: $X = Y_0 \supset Y_1 \supset \cdots \supset Y_n = \{Pt\}$.
- Let D be a big divisor on X. By looking at successive vanishing orders along the elements of the flag, we get a function:

$$\nu : H^0(X, \mathcal{O}_X(D)) \smallsetminus \{0\} \longrightarrow \mathbf{Z}^n \subset \mathbf{R}^n$$
$$s \longmapsto (c_1, \ldots, c_n)$$

  Let us denote the image by $\nu(D)$.
- Define the Okounkov Body $\Delta(D)$ as:

$$\Delta(D) = \Delta(D)_{Y_\bullet} =_{def} \text{closed convex hull} \left(\bigcup_{m \geq 1} \frac{1}{m} \cdot \nu(mD) \right).$$

Example: X smooth toric variety, $\{Y_\bullet\}$ flag of T-invariant subvarieties, D T-invariant divisor $\Longrightarrow \Delta(D)$ is equal to P_D up to translation.

Main properties:

- For each big divisor D on $X \Longrightarrow$ Get a convex compact set $\Delta(D) \subseteq \mathbf{R}^n$, that depends only on the numerically equivalence class of D, and satisfies

$$n! \cdot \mathrm{vol}_{\mathbf{R}^n}\big(\Delta(D)\big) = \mathrm{vol}_X(D) =_{\mathrm{def}} \lim_{m \to \infty} \frac{h^0(X, \mathcal{O}_x(mD))}{m^n / n!}.$$

- Global Okounkov body of X: There exists a closed convex cone $\Delta(X)$, such that in the diagram

$$\Delta(X) \hookrightarrow \mathbf{R}^n \times \mathbf{N}^1(\mathbf{X})_{\mathbf{R}} \to N^1(X)_{\mathbf{R}},$$

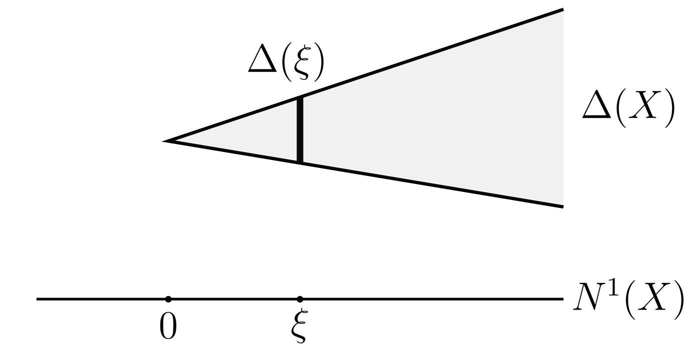


Figure 1. The global Okounkov body.

the fiber $\Delta(X)_D \subseteq \mathbf{R}^d \times \{\mathbf{D}\} = \mathbf{R}^n$ of $\Delta(X)$ over any big numerical class $D \in N^1(X)_{\mathbf{Q}}$ is $\Delta(D)$.

Answer to the first question: Let $\mathbf{P}(\mathcal{E})$ be the projectivization of a rank two TVB $\mathcal{E}$ over a smooth projective TV X.

- There is a natural flag of T-invariant subvarieties on $\mathbf{P}(\mathcal{E})$.

$$\mathbf{P}(\mathcal{E}) \supset \mathbf{P}(\mathcal{E}|_{Y_1}) \supset \cdots \supset \mathbf{P}(\mathcal{E}|_{Y_n}) \supset \{pt\}.$$

- The Global Okounkov Body of $\mathbf{P}(\mathcal{E})$ can be explicitly described as a polyhedral cone in $\mathbf{R}^{n+1} \times \mathbf{N}^1(\mathbf{P}(\mathcal{E}))_{\mathbf{R}}$.
- In the proof, we give a finite collection of linear inequalities $\{I_J\}_{J \subseteq \{1,\ldots,d\}}$, such that

$$\Delta\big(\mathbf{P}(\mathcal{E})\big) = \Big\{ I_J(\mathbf{x}) \leq 0 \colon \mathbf{x} \in \mathbf{R}^{n+1} \times \mathbf{N}^1(\mathbf{P}(\mathcal{E}))_{\mathbf{R}}, \mathbf{J} \subseteq \{\mathbf{1}, \ldots, \mathbf{d}\} \Big\}.$$

What do our linear inequalities look like?

$\mathcal{E}$ TVB $\Longrightarrow$ Get $\{a_j\}_{1 \leq j \leq d}$, $\{b_j\}_{1 \leq j \leq d} \subseteq \mathbf{Z}$; $u_1, u_2 \in \mathbf{Z}^n$; each filtration is classified as one of three types A, B, C; combinatorial data from TV X.

Identify: $\mathbf{R}^{n+1} \times \mathbf{N}^1(\mathbf{P}(\mathcal{E}))_{\mathbf{R}} \cong \mathbf{R}^{d+2}_{(\mathbf{x}_1,\ldots,\mathbf{x}_n,\mathbf{x}_{n+1},\mathbf{w}_{n+1},\ldots,\mathbf{w}_d,\mathbf{w})}$.

THEOREM The following inequalities define the global Okounkov body $\Delta(\mathbf{P}(\mathcal{E}))$ of $\mathbf{P}(\mathcal{E})$: $w \geq x_{n+1} \geq 0$ and

$$\sum_{i=1}^{n} \langle v_i^*, v_j \rangle x_i + \langle u_2 - u_1, v_j \rangle x_{n+1} + (a_j - \langle u_2, v_j \rangle) w + w_j \geq 0, \ \forall\, j \in A,$$

$$\sum_{i=1}^{n} \langle v_i^*, v_j \rangle x_i + (\langle u_2 - u_1, v_j \rangle + b_j - a_j) x_{n+1} + (a_j - \langle u_2, v_j \rangle) w + w_j \geq 0, \ \forall\, j \in B,$$

$$\sum_{j \in J} \frac{1}{b_j - a_j} \left[\sum_{i=1}^{n} \langle v_i^*, v_j \rangle x_i + \langle u_2 - u_1, v_j \rangle x_{n+1} + (a_j - \langle u_2, v_j \rangle) w + w_j \right] + w - x_{n+1} \geq 0,$$

for each admissible set $J \subseteq C$.

Example: The tangent bundle $T_{\mathbf{P}^2}$ of $\mathbf{P}^2$. The Okounkov body of $\mathcal{O}_{\mathbf{P}(T_{\mathbf{P}^2})}(1)$ inside $\mathbf{R}^3_{(\mathbf{x},\mathbf{y},\mathbf{z})}$ is given by the linear inequalities:

$$x \leq 1 \qquad y \geq 0 \qquad x + y + z \leq 2$$
$$x \geq 0 \qquad z \geq 0 \qquad z \leq 1$$

In particular, $\mathrm{vol}\Big(\mathcal{O}_{\mathbf{P}(T_{\mathbf{P}^2})}(1)\Big) = 3!$.

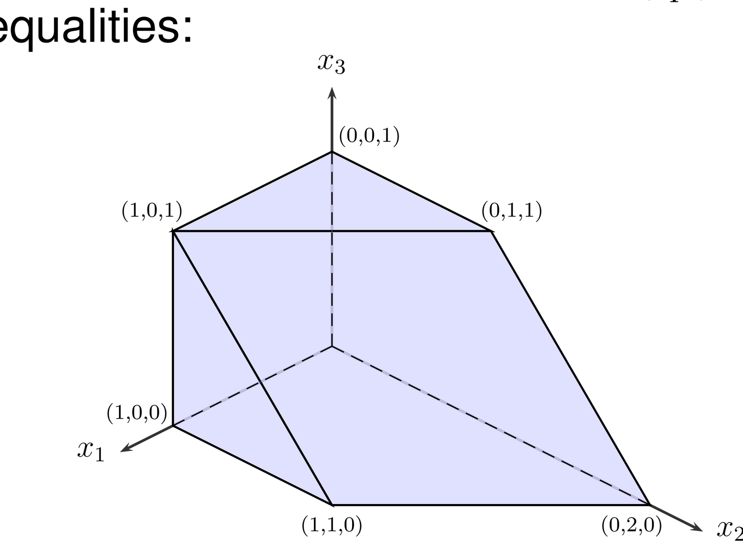


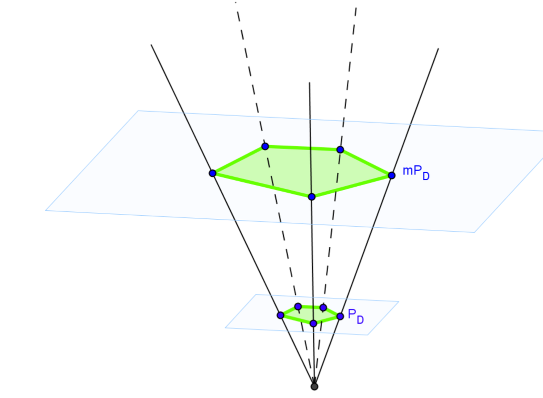
Figure 2. The Okounkov body of $\mathcal{O}_{\mathbf{P}(T_{\mathbf{P}^2})}(1)$.

4 Mori dream spaces [Hu-Keel]

A normal projective $\mathbf{Q}$-factorial variety X is a *Mori Dream Space* if $\mathrm{Pic}(X)_{\mathbf{Q}} = N^1(X)_{\mathbf{Q}}$, and for some (or equivalently any) line bundles $L_1, \ldots, L_r$ that generate $\mathrm{Pic}(X)$, the associated *Cox Ring* is a finitely generated k-algebra:

$$\mathrm{Cox}(X, L_1, \ldots, L_r) := \bigoplus_{(m_1,\ldots,m_r) \in \mathbf{Z}^r} H^0(X, L_1^{\otimes m_1} \otimes \cdots \otimes L_r^{\otimes m_r}).$$

Example: Toric varieties are Mori dream spaces.



The finite generation of $R = \bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mD))$ is equivalent to the finite generation of the semigroup of lattice points in the cone.

[Elizondo].

THEOREM $\mathbf{P}(\mathcal{E})$ is a Mori Dream Space, for any rank two toric vector bundle $\mathcal{E}$ over a simplicial projective toric variety.

Remark: F. Knop (1993) and J. Hausen and H. Suss (2009), prove the finite generation of Cox rings in the more general setting of complexity-one T-varieties. Our methods are independent from both of these works, and are inspired by the question of the finite generation of the Cox rings of higher rank toric vector bundles.

The proof of the first theorem in few words:

- We describe the Klyachko filtrations of tensor products and symmetric products (more generally of Schur functors) of TVB. With this, we get descriptions of the spaces of sections of all line bundles on $\mathbf{P}(\mathcal{E})$.
- We construct a special collection of equivariant sections for each line bundle on $\mathbf{P}(\mathcal{E})$. We give a formula for the images of these sections under ν, and show that the closed convex hull of these images is $\Delta(\mathbf{P}(\mathcal{E}))$. By translating the conditions for the existence of these sections into linear inequalities, we get the description of $\Delta(\mathbf{P}(\mathcal{E}))$.

Two proofs of the second theorem in few words:

- The torus action induces a finer grading in $\mathrm{Cox}(\mathbf{P}(\mathcal{E}))$. We describe the graded pieces and the multiplication map in terms of the data arising from the Klyachko filtrations of $\mathcal{E}$. We deal with some features not present in the TV case, and exhibit a finite generator set of a Veronese subalgebra of $\mathrm{Cox}(\mathbf{P}(\mathcal{E}))$.
- An appropriate Veronese subalgebra of $\mathrm{Cox}(\mathbf{P}(\mathcal{E}))$ is isomorphic to the semigroup algebra obtained from the semigroup of lattice points with sufficiently divisible coordinates in $\Delta(\mathbf{P}(\mathcal{E}))$, which is finitely generated using the first theorem.