GIT Compactifications of $\mathcal{M}_{0,n}$ from Conics

Noah Giansiracusa (joint work with Matthew Simpson)
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1 Introduction

We study a family of GIT quotients parametrizing n-pointed conics that generalize the GIT quotients $(\mathbb{P}^1)^n//\mathrm{SL}_2$. These latter quotients compactify the moduli space $\mathcal{M}_{0,n}$ of nonsingular n-pointed rational curves by allowing points to collide as long as their *weight* (a number assigned to each point by the GIT linearization) is not too much. For the quotients we investigate, denoted $\mathtt{Con}(n)//\mathrm{SL}_3$, the compactification allows some points to overlap but if their weight is too great then the nonsingular conic degenerates into a nodal conic. Up to isomorphism nonsingular and nodal conics are a $\mathbb{P}^1$ and a pair of intersecting $\mathbb{P}^1$s, respectively, so the spaces $\mathtt{Con}(n)//\mathrm{SL}_3$ can be viewed as intermediate compactifications between $(\mathbb{P}^1)^n//\mathrm{SL}_2$ and the Deligne-Mumford-Knudsen compactification $\overline{\mathcal{M}}_{0,n}$.

Our main result is that $\overline{\mathcal{M}}_{0,n}$ maps to all possible GIT quotients $\mathtt{Con}(n)//\mathrm{SL}_3$, and that many of these morphisms factor through Hassett's spaces $\overline{\mathcal{M}}_{0,\vec{c}}$ of weighted pointed curves.

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2 GIT Stability of n-pointed conics

The following theorem, proven using the Hilbert-Mumford numerical criterion, characterizes GIT stability for pointed conics.

Theorem 2.1. *Let* $\vec{w} = (\gamma, c_1, \ldots, c_n)$ *specify an ample fractional line bundle on the space of* n*-pointed conics* $\mathtt{Con}(n) \subset \mathbb{P}(\mathrm{Sym}^2(V^*)) \times (\mathbb{P}(V))^n$, $V = \mathbb{C}^3$, *linearized for the natural action of* $\mathrm{SL}(V)$. *If* $c := c_1 + \cdots + c_n$ *then:*

- *all non-reduced conics are unstable*
- *a nodal conic is semistable iff*
  1. *the weight of marked points at any smooth point is* $\leq \frac{c+\gamma}{3}$
  2. *the weight of marked points at the node is* $\leq c - 2(\frac{c+\gamma}{3})$, *and*
  3. *the weight on each component away from the node is* $\geq \frac{c+\gamma}{3}$
- *a nonsingular conic is semistable iff the weight at each point is* $\leq \min\{\frac{c+\gamma}{3}, \frac{c}{2}\}$

In particular, if $\gamma > \frac{c}{2}$ *then nodal conics are unstable. Stability of nodal and nonsingular conics is characterized by the corresponding inequalities being replaced by strict inequalities.*

3 Variation of GIT

When a reductive group G acts on a variety the space of linearized fractional polarizations forms a cone called the G*-ample cone*, and inside it sits the G*-effective cone* which is defined as the set of linearizations for which the semistable locus is nonempty. The G-effective cone admits a finite wall and chamber decomposition such that on each open chamber the GIT quotient is constant and when a wall is crossed the quotient undergoes a birational modification (see [Tha96] and [DH98]). In some cases this cone admits a natural cross-section so that the (closure of) the space of linearizations can be identified with a certain polytope which we call the *linearization polytope*.

For example, the linearization polytope for the GIT quotients $(\mathbb{P}^m)^n//\mathrm{SL}_{m+1}$ parametrizing configurations of n points in $\mathbb{P}^m$ is the hypersimplex

$$\Delta(m+1, n) = \{\vec{w} \in \mathbb{Q}^n \mid 0 \leq w_i \leq 1, \sum_{i=1}^{n} w_i = m+1\}$$

with walls of the form $\sum_{i \in I} w_i = k$ for $I \subset \{1, \ldots, n\}$ and $1 \leq k \leq m$ (see, e.g., Example 3.3.21 in [DH98]). In particular, for points on the line $(m = 1)$ we have $\Delta(2, n)$ with walls $\sum w_i = 1$, and for points in the plane $(m = 2)$ we have $\Delta(3, n)$ with walls $\sum w_i = 1$ and $\sum w_i = 2$. A consequence of Theorem 2.1 is that for the space of n-pointed conics the effective linearizations form a 1-parameter family of hypersimplices that interpolate these two cases.

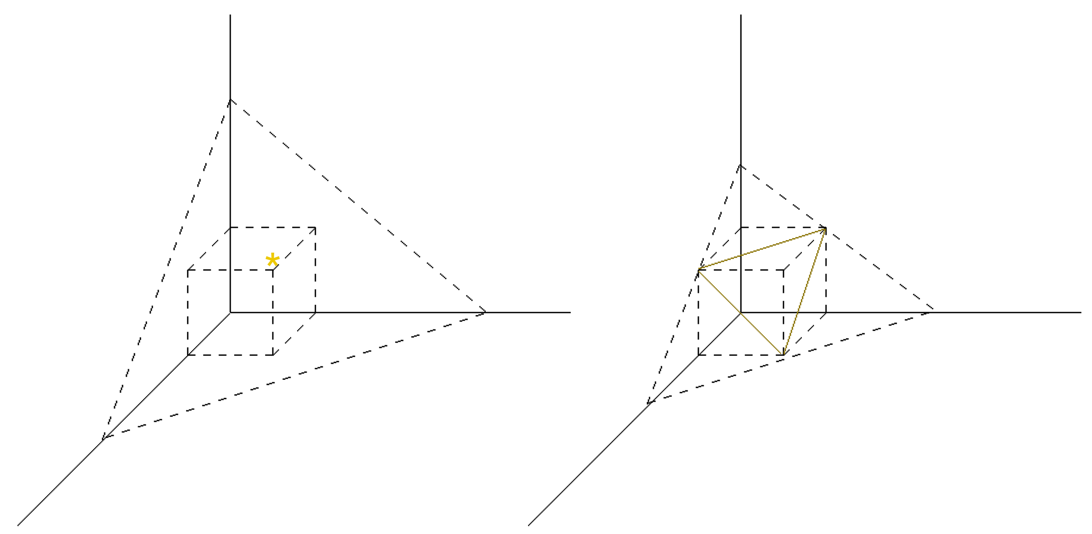

Figure 1: Hypersimplices $\Delta(3, 3)$ (left) and $\Delta(2, 3)$ (right).

Corollary 3.1. *The* SL_3*-effective cone on* $\mathtt{Con}(n)$ *induced from that of the ambient* $\mathbb{P}^5 \times (\mathbb{P}^2)^n$ *is subdivided by the hyperplane* $\gamma = \frac{c}{2}$ *into two subcones:* $\gamma \leq \frac{c}{2}$ *for which semistable nodal conics occur, and* $\gamma > \frac{c}{2}$ *for which singular conics are unstable. With cross-sections* $\gamma + c = 3$ *on the former and* $c = 2$ *on the latter the linearization polytopes for fixed* γ *are* $\Delta(3 - \gamma, n)$ *with walls* $\sum c_i = 1$ *and* $\sum c_i = 2$ *if* $0 \leq \gamma \leq 1$, *and* $\Delta(2, n)$ *with walls* $\sum c_i = 1$ *if* $\gamma \geq 1$.

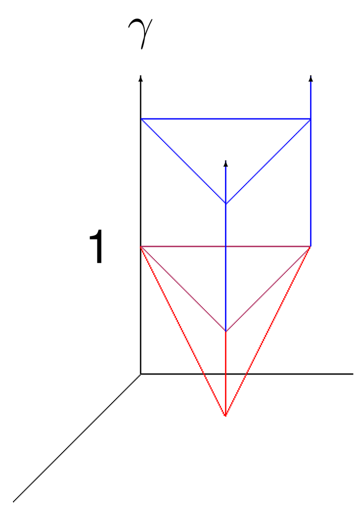

Figure 2: The space of linearizations for $\mathtt{Con}(3)$.

Remark 3.2. If $\gamma \geq \frac{c}{2}$ then $\mathtt{Con}(n)//_{(\gamma,\vec{c})}\mathrm{SL}_3 \cong (\mathbb{P}^1)^n//_{\vec{c}}\,\mathrm{SL}_2$ so we can restrict γ to the interval $[0, \frac{c}{2}]$, in which case we only need one normalization (namely $\gamma + c = 3$) and the linearization polytope is $\Delta(3, n+1)$ with walls $\sum w_i = 1$ and $\sum w_i = 2$ which are "vertical" in the sense that they are independent of γ.

4 Hassett's weighted pointed curves

Another interesting family of compactifications is provided by Hassett's moduli spaces of stable weighted pointed curves [Has03]. For a weight vector $\vec{c} \in [0, 1]^n$ the space $\overline{\mathcal{M}}_{0,\vec{c}}$ parametrizes nodal rational curves with marked points p_i avoiding the nodes such that on any component C we have $\sum_{p_i \in C} c_i + \delta_C > 2$, where δ_C is the number of nodes on C. In particular, if $c_i = 1$ for $1 \leq i \leq n$ then $\overline{\mathcal{M}}_{0,\vec{c}} \cong \overline{\mathcal{M}}_{0,n}$ so these spaces can also be viewed as intermediate compactifications of $\mathcal{M}_{0,n} \subset \overline{\mathcal{M}}_{0,n}$. These Hassett compactifications and our conic compactifications are related in the following manner.

Theorem 4.1. *For any* $\vec{w} = (\gamma, \vec{c}) \in \Delta(3, n+1)$ *there is a morphism* $\overline{\mathcal{M}}_{0,n} \to \mathtt{Con}(n)//_{\vec{w}}\,\mathrm{SL}_3$. *If* $\vec{w}$ *lies in the interior of a GIT chamber then this morphism factors through Hassett's space* $\overline{\mathcal{M}}_{0,\vec{c}}$.

Our choice of normalization forces $\sum_{i=1}^{n} c_i < 3$, which in turn forces all stable curves parametrized by $\overline{\mathcal{M}}_{0,\vec{c}}$ to be *chains* of $\mathbb{P}^1$s. The inner components of these chains can then be contracted to produce a nodal conic which turns out to be stable with respect to the GIT linearization corresponding to the Hassett weight data.

Remark 4.2. This theorem should be thought of as an analogue of the result of Kapranov [Kap93] that $\overline{\mathcal{M}}_{0,n}$ admits a morphism to every GIT quotient $(\mathbb{P}^1)^n//\mathrm{SL}_2$. In fact, because $\mathtt{Con}(n)//_{(\gamma,\vec{c})}\mathrm{SL}_3 \cong (\mathbb{P}^1)^n//_{\vec{c}}\,\mathrm{SL}_2$ for $\gamma > 1$, this theorem when combined with Kapranov's result shows that $\overline{\mathcal{M}}_{0,n}$ admits a morphism to every GIT quotient $\mathtt{Con}(n)//\mathrm{SL}_3$.

5 Semistable reduction

The morphism described in Theorem 4.1 can be used to study semistable reduction in the spaces $\mathtt{Con}(n)//\mathrm{SL}_3$. Any 1-parameter family of semistable configurations of n points on a conic must have a semistable limit since the GIT quotient is proper. If the conics are nonsingular then we can identify them with $\mathbb{P}^1$ and the limit may be computed by first finding the limit as a stable curve in $\overline{\mathcal{M}}_{0,n}$ and then looking at the image of this curve under the morphism $\overline{\mathcal{M}}_{0,n} \to \mathtt{Con}(n)//\mathrm{SL}_3$.

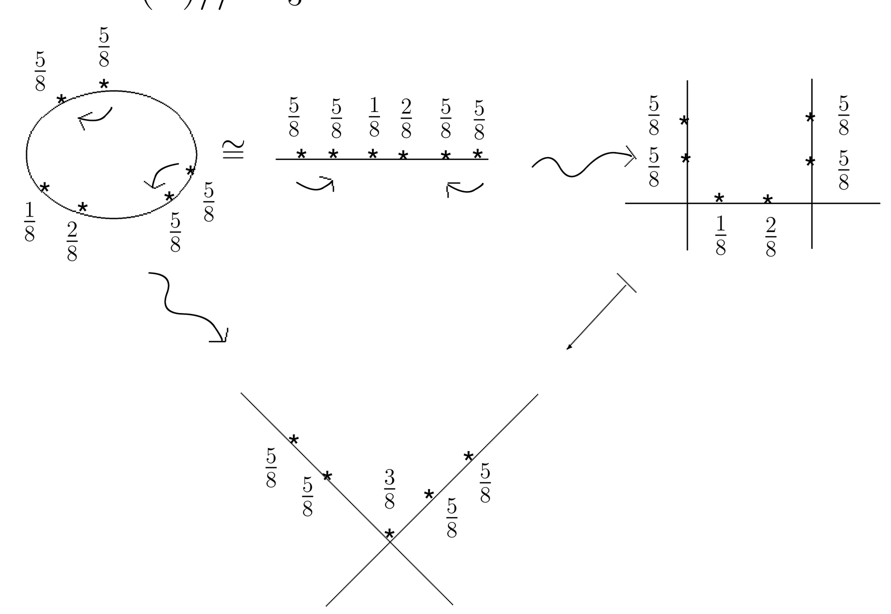

Figure 3: An example of semistable reduction with $\gamma = \frac{1}{8}, \vec{c} = (\frac{5}{8}, \frac{5}{8}, \frac{5}{8}, \frac{5}{8}, \frac{2}{8}, \frac{1}{8})$.

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