PROGRESS IN NAGATA'S CONJECTURE FOR TEN POINTS

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1. Interpolation Problems in $\mathbb{P}^2$

Notation

- Fix general points $p_1, \dots, p_n$ in the projective space $\mathbb{P}^r$, and multiplicities $m_1, \dots, m_n$.Let $\mathcal{L} = \mathcal{L}_{r,d}(m_1, \dots, m_n)$ denote the linear system of hypersurfaces C of degree d in $\mathbb{P}^r$ having multiplicity at least m_i at p_i for each i: $\mathrm{mult}_{p_i}(C) \geq m_i$.
- In algebraic settings $\mathcal{L}$ consists of homogeneous polynomials in $r+1$ variables of degree d for which all partial derivatives up to order $m_i - 1$ vanish at the points p_i. We will consider the planar ($r = 2$) case for homogeneous linear systems and we will write it as $\mathcal{L}_d(m^n)$.

If $d = 4$, $m = 2$, $n = 5$ then $\mathcal{L}_4(2^5)$ is the (projective) space of plane quartics with five double points.

Virtual and Expected Dimensions

The space of plane hypersurfaces of degree d in $\mathbb{P}^n$ has projective dimension $\binom{d+r}{r} - 1$.

Imposing a point of multiplicity m is $\binom{m+r-1}{r}$ conditions.

The projective virtual dimension of $\mathcal{L}$ is

- $v(\mathcal{L}_{r,d}(m^n)) = \binom{d+r}{r} - n\binom{m+(r-1)}{r} - 1$ and the projective expected dimension is
- $e(\mathcal{L}) = \max\{-1, v\}$
- -1 means an empty projective space, a zero vector space

The Blowup of the Plane

- For the planar case of linear system the virtual dimension is $v(\mathcal{L}) = d(d+3)/2 - nm(m+1)/2$
- Consider the blowup of $\mathbb{P}^2$ at the r points: This creates a surface X, with divisor classes H the class of a line and $E_1, \dots, E_r$ exceptional divisors. The relevant sheaf is $L = \mathcal{O}_X(dH - \sum_i mE_i)$.

$$v = \dim H^0(X, L) - \dim H^1(X, L) - 1$$

Having the expected dimension: either $H^0 = 0$ or $H^1 = 0$.

- If all the conditions imposed are independent then the linear system has the expected dimension and therefore

$$\dim(\mathcal{L}) \geq \text{expected} \geq \text{virtual}$$

Naive Conjecture

Naive Conjecture:
$\mathcal{L} = \mathcal{L}_d(m^r)$ has the expected dimension.
This is wrong:

- 1. The linear system $\mathcal{L}_2(2^2)$ has a negative virtual dimension $v = 5 - 2 \cdot 3 = -1$ but is not empty since it consists of the double line through the two points.
- 2. Another example: $\mathcal{L}_4(2^5)$ consists of the double conic through the five points that is again non empty even though $v = 14 - 5 \cdot 3 = -1$

Note that in examples 1 and 2, the linear systems $\mathcal{L}_2(2^2)$ and $\mathcal{L}_4(2^5)$ contained a base locus consisting of a multiple curve.

Other Related Conjectures

If there exists a (-1)-curve C (on the blowup X of the plane) such that

$$\mathcal{L} \cdot C \leq -2$$

and

$$\dim(\mathcal{L}) \geq 0$$

then $\mathcal{L}$ does not have the expected dimension. The curve C will split at least twice from the system, and the virtual dimension of the residual goes up.

GHH Conjecture:
If no such (-1)-curve exists, then $\mathcal{L}$ has the expected dimension.

Nagata's Conjecture: (for $\mathcal{L}_d(m^n)$) If $n > 9$ and $d^2 < nm^2$ then $\mathcal{L}_d(m^n)$ is empty.

Partial progress

- Nagata's conjecture is true if n is a perfect square, and if $n < 10$, so the number ten appears to be the boundary case.
- For small values of n Harbourne-Roé and Dumnicki proved the emptiness of the linear system for some rational approximations of $\sqrt{n}$.

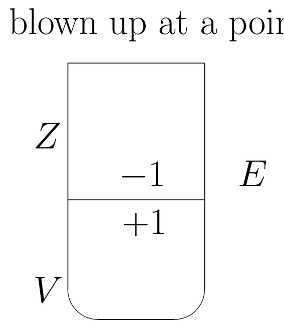
Nagata's conjecture for ten points: If $\frac{d}{m} < \sqrt{10} \cong 3.1622$ then $\mathcal{L}_d(m^{10})$ is empty.

- For ten points Ciliberto and Miranda used degenerations of the plane to prove the non-speciality of $\mathcal{L}_d(m^{10})$ for $d/m \geq \frac{174}{55} = 3.1636..$, approaching $\sqrt{10}$ by rational numbers from above.
- Approaching $\sqrt{10}$ by rational numbers from below, i.e. for $d/m < 117/37 \cong 3.1621..$ we prove the emptiness of the linear system using the same degeneration.

2. Three Degenerations of the plane

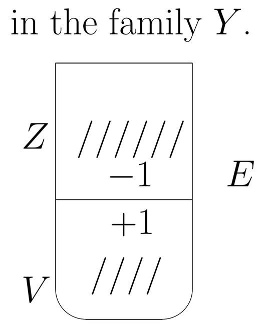
The Degeneration of the plane

- We will degenerate both of the plane (blown up at the ten general points) and the line bundle.
- We first consider the trival family of $\mathbb{P}^2$ over a disc and we blow up a point in the central fiber.
- We will get a new family where the general fiber is a $\mathbb{P}^2$, and the central fiber is the union of two surfaces $V \cup F$, where $P \cong \mathbb{P}^2$ is a projective plane, $Z \cong \mathbb{F}_1$ is a plane blown up at a point.



The Degeneration of the blown-up plane

We now choose four general points on V and six general points on Z and we consider these ten points as limits of ten general points in the general fiber and we blow these points up in the family Y.



- V is a plane blown up at four general points;
- Z is a plane blown up at seven general points;
- V and Z meet along a smooth rational curve E

Pulling back $\mathcal{L}_d(m^{10})$ to the central fiber we get $\mathcal{L}_0(m^4)$ on V and $\mathcal{L}_d(0, m^6)$ on Z.

Central Effectivity

- Consider all the possible limits of $\mathcal{L}_d(m^{10})$ obtained by twisting by a multiple of Z. Namely, if a is the parameter, the linear systems will have the form

$$\mathcal{L}_V = \mathcal{L}_a(m^4), \quad \mathcal{L}_Z = \mathcal{L}_d(a, m^6).$$

- If $\mathcal{L}_d(m^{10})$ is nonempty then there is a limit curve in the central fiber and therefore both the restrictions to V and Z are nonempty for some value of the parameter a.
- Conversely, suppose there is a ratio d/m such that for any value of the parameter a at least one of the linear systems $\mathcal{L}_V$ and $\mathcal{L}_Z$ is empty. Then there is no limit curve in central fiber and therefore $\mathcal{L}_d(m^{10})$ is empty. We will exploit this idea for every degeneration that we consider.

Lemma 1. *If $d/m < 3$ then $\mathcal{L}_d(m^{10})$ is empty.*

Proof.

- If $a < 2m$ then $\mathcal{L}_V = \mathcal{L}_a(m^4)$ is empty. Indeed, if $a = 2m$ the linear system is composed of a pencil and consists of m conics through the four points. If $a < 2m$, it is empty.
- If $a \geq 2m$ then $\mathcal{L}_Z = \mathcal{L}_d(a, m^6)$ is empty. In $\mathcal{L}_Z$ the cubic $\mathcal{L}_3(2, 1^6)$ splits off $-3d + 6m + 2a$ times

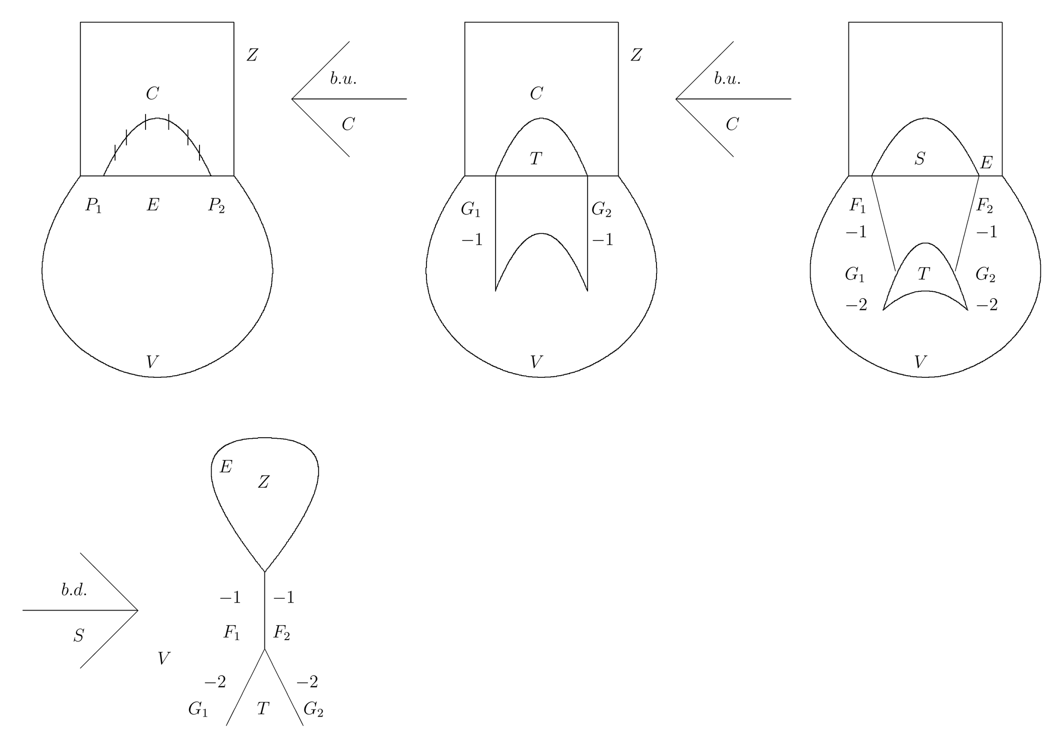
$$\mathcal{L}_Z = (6m + 2a - 3d)\mathcal{L}_3(2, 1^6) + Res$$

The residual system $Res = \phi$ since its degree is negative. □

We notice that if $d/m = 3$ and $a = 2m$ both of the linear systems are nonempty.
Indeed, $\mathcal{L}_Z$ consists of a multiple cubic $\mathcal{L}_Z = m\mathcal{L}_3(2, 1^6)$ and $\mathcal{L}_V = \mathcal{L}_{2m}(m^4)$ consists of a pencil of conics.

The 2-throw

The cubic C intersects E twice, we will blow it up twice on Z and then contract S

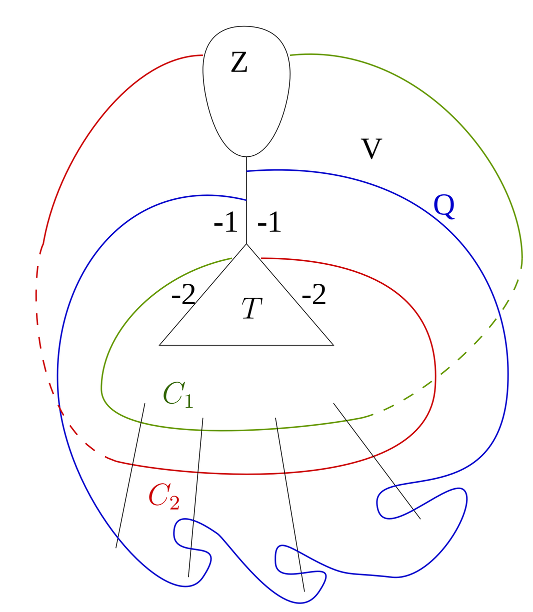


The second degeneration

Lemma 2. *If $\frac{d}{m} < 3.15$ then $\mathcal{L}_d(m^{10})$ is empty.*

Proof. One of the three linear systems $\mathcal{L}_T$, $\mathcal{L}_Z$ or $\mathcal{L}_V$ is empty no matter what limit bundle you have. □

Denote by Q_i the four quartics on V, of the form
$Q_1 = \mathcal{L}_4(2^3, 1, [1,1]^2)$ and by C_i the two conics
$C_1 = \mathcal{L}_2(1^4, [1,0], [0,0])$ and $C_2 = \mathcal{L}_2(1^4, [0,0], [1,0])$.
Q_i and C_i are all disjoint -1 curves.
If $d/m = 3.15$ all three linear system are nonempty and $\mathcal{L}_V$ is special consisting of six -1 curves: C_i and Q_i, so we will throw them.



The third degeneration

Theorem 3. *If we fix d and m then, all limits of the linear system $\mathcal{L}_d(m^{10})$ are of the following form for some integer values of the parameters z_i, q, x, y and e*

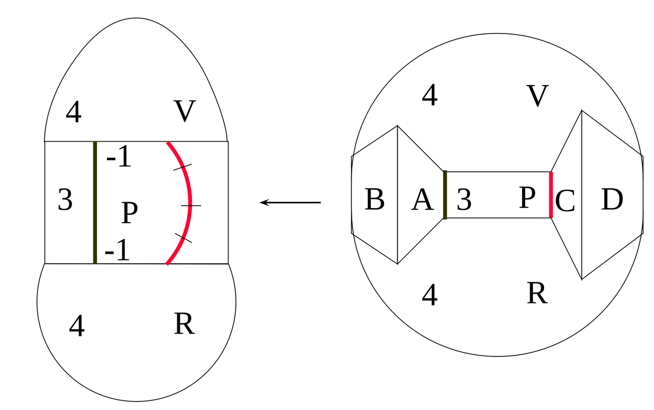
- $\mathcal{L}_Z = \mathcal{L}_{3q-3m+d}(q^6, [x, e-x], [y, e-y])$
- $\mathcal{L}_T = \mathcal{L}_{-2q-16m+5d+2e}([x, e-x], [y, e-y])$
- $\mathcal{L}_V = \mathcal{L}_{-q-41m+13d}([-2q-16m+5d+e, 0]^2, [z_i, -6d+19m-z_i]^2_{i\, i=1,..,4})$
- $\mathcal{L}_{U_1} = \mathcal{L}_{2x-e}$
- $\mathcal{L}_{U_2} = \mathcal{L}_{2y-e}$
- $\mathcal{L}_{Y_i} = \mathcal{L}_{2z_i-19m+6d}$.

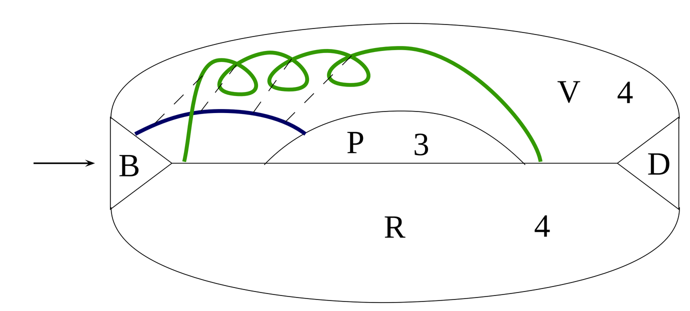
Corollary 4. *If $d/m < \frac{117}{37}$ then $\mathcal{L}_d(m^{10})$ is empty.*

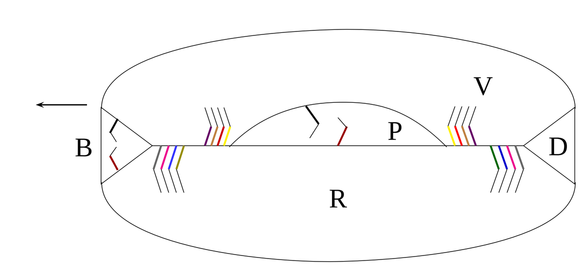
3. 11 and 12 points

11 points

- Blow up two points on $P = \mathbb{P}^2$ and consider a degeneration of the plane obtained by blowing up the exceptional divisors. We get two more planes V and R, and we will put 4-3-4 points on each.
- In the first degeneration a line and a conic will split off on P so we will throw them. This creates two more planes B and D.
- In the second degeneration, a conic and four quartics are fixed divisors on V and on R so we will throw them again creating 10 more planes.
- From the third degeneration we will conclude the emptiness of $\mathcal{L}_d(m^{11})$ for $d/m < 3.3148$. Note also that $\sqrt{11} = 3.3166..$







12 points Consider a degeneration of the central fiber into a union of $P = \mathbb{P}^2$ and $F = \mathbb{F}_1$ and put eight points on F and four poins on P. Note that $\sqrt{12} = 3.464..$

Theorem 5. *If $d/m < 3.4594$ then $\mathcal{L}_d(m^{11})$ is empty.*

In this case, eight curves $\mathcal{L}_{13}(7, 4^7, 3)$ and one quartic $\mathcal{L}_4(3, 1^8)$ split off on F.