Stable maps relative to simple normal crossing divisors

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1 Introduction

The idea of logarithmic stable maps was introduced by Bernd Siebert in 2001. But the program has been on hold for a while, since Mark Gross and Bernd Siebert were working on other projects in mirror symmetry. They have returned to it only recently. Their approach is to probe the moduli space using the standard log point. Here we describe our alternative approach. This covers the case where the target is a variety with a simple normal crossings divisor.

Inspired by the work of B. Kim on logarithmic stable maps, our starting point is the notion of minimal log structures. However, instead of using expensions as in B. Kim's situation, we put minimal log structures on the source curves, but fix the target with the canonical log structure associated to the simple normal crossing divisors. Our log stable maps will be the usual stable maps equipped with the minimal log structures. The stack parametrizing log stable maps under our definition is shown to be a proper Deligne-Mumford stack. This is in part joint work with Dan Abramovich.

It turns out that our minimal structure for the relative case can be slightly modified for the case of degenerate target. We are currently working on an explicit description of this case. It should be pointed out that our construction does not cover the case of normal crossing divisors which are not simple, but we hope one could use descent arguments to figure this out.

2 Logarithmic structures

Let $\underline{X}$ be a scheme. A *log structure* on $\underline{X}$ is a pair $(\mathcal{M}, \exp)$, which consists of a sheaf of monoids $\mathcal{M}$ on the étale site $\underline{X}_{\acute{e}t}$ of $\underline{X}$, and a morphism of sheaves of monoids $\exp : \mathcal{M} \to \mathcal{O}_{\underline{X}}$, such that $\exp^{-1}(\mathcal{O}^*_{\underline{X}}) \cong \mathcal{O}^*_{\underline{X}}$ via $\exp$. Here we view $\mathcal{O}_{\underline{X}}$ as a monoid under multiplication. We call the pair $(\underline{X}, \mathcal{M})$ a *log scheme*. A good example of log structures given by K. Kato is the following:

Example 1. Let $\underline{X}$ be a smooth scheme, and D a normal crossing divisor on $\underline{X}$. Then there is a canonical log structure $\mathcal{M}_X$ associated to D given by

$$\mathcal{M}_X := \{\, f \in \mathcal{O}_{\underline{X}} \mid f \text{ is invertible outside } D\} \subset \mathcal{O}_{\underline{X}}.$$

Throughout this poster, we fix a projective smooth scheme $\underline{X}$, and a simple normal crossing divisor $D = \cup_{i=1}^{k} D_i$, where D_i is a smooth divisor for all i. Let $\mathcal{M}_X$ be the log structure associated to D on $\underline{X}$. Denote by X the log scheme $(\underline{X}, \mathcal{M}_X)$, which will be our target of log maps.

3 A categorical description of minimality

Definition 2. A log map over a fs (fine and saturated) log scheme S is a tuple $\xi = (C \to S, f)$, where

1. The source $C \to S$ is a family of log smooth curves, whose underlying $\underline{C} \to \underline{S}$ is a usual pre-stable curve.
2. $f : C \to X$ is a map of log schemes.

Given another log map $\xi' = (C' \to S', f')$, an arrow $\xi' \to \xi$ is given by the following cartesian diagram of log schemes:

$$\begin{array}{ccccc} C' & \to & C & \xrightarrow{f} & X^{log} \\ \downarrow & & \downarrow & & \\ S' & \to & S & & \end{array}$$

such that the composition of the top arrows is f'.

Inspired by the work of F. Kato on the canonical log structure on curves, we ask the following questions:

Question 3. *For any log map ξ over S as above, does there exists a unique (up to a unique isomorphism) log map ξ_{min} with the same underlying structure $(\underline{C} \to \underline{S}, \underline{f})$, such that*

1. *There is a unique (up to a unique isomorphism) arrow $g : \xi \to \xi_{min}$.*
2. *Any arrow $\xi_{min} \to \xi$ is an isomorphism of log maps.*

We call such ξ_{min} a minimal log map over $\underline{S}$.

We give an affirmative answer to this question by construct the minimal log maps as following.

4 An explicit description of minimality

We start from a geometric fiber. Consider a log map $\xi = (C \to S, f)$ such that $\underline{S}$ is a geometric point. We associate to ξ a weighted oriented graph G as follows:

1. The underlying graph $\underline{G}$ is the dual graph of the curve $\underline{C}$.
2. The weight on each vertex v is a k-tuple $(e_{v,i})_{i=1}^{k}$, such that $e_{v,i} = 0$ if the irreducible component corresponding to v does not map into D_i.
3. The weight on each edge l is a variable e_l and a k-tuple of positive integers $(c_{l,i})_{i=1}^{k}$.
4. There exist k possibly different orientations on G, denote by a set of partial orders on the set of vertices $\{\leq_i\}_{i=1}^{k}$.

Consider an edge l joining two vertices v and v'. We assume that $v \leq_i v'$, namely the i-th orientation of l is from v to v'. Then we put the condition

$$e_{v',i} = e_{v,i} + c_{l,i} \cdot e_l.$$

With those conditions, the weights $e_{v,i}$ and e_l generate a monoid, denoted by $\overline{M}$. Let $\overline{M}(G)$ be the saturation of $\overline{M}$. In fact, all the information about the weights and orientations is coming from the log map f. Roughly speaking, the monoid $\overline{M}(G)$ carries the "minimal requirements" that a log map should satisfy.

Definition 4. With the notations as above, ξ is minimal if $\overline{\mathcal{M}}_S \cong \overline{M}(G)$. A log map ξ_T over T is called minimal, if for each geometric point $\bar{t} \in \underline{T}$, the fiber $\xi_{T,\bar{t}}$ is minimal.

Theorem 5. *The log map ξ_{min} and the arrow g in question 3 exists, and the data (ξ_{min}, g) is unique up to a unique isomorphism. Furthermore, the log map ξ_{min} is minimal as in the definition 4.*

5 Logarithmic stable maps

We use the symbol Γ to denote the discrete data including n-markings, genus g, the curve class β in $\underline{X}$, and the tangency multiplicities for each marked points tangent to D_i for each i.

Definition 6. A log stable map over $\underline{S}$ is a minimal log map $\xi = (C \to S, f)$, such that the underlying structure is a usual stable map. It is called a Γ-log stable map, if it satisfies the condition given by the discrete data Γ.

Denote by $\mathcal{K}_\Gamma(X)$ the fibered category of log stable maps over the category of schemes. It is not hard to see that this is a stack. By theorem 5, we can also view $\mathcal{K}_\Gamma(X)$ as a log stack over the category of fs log schemes, which parametrizes log maps whose underlying structure is stable in the usual sense.

Theorem 7. *$\mathcal{K}_\Gamma(X)$ is a proper Deligne-Mumford stack.*

We give a sketch of the proof as follows:

1. *Algbricity.* The deformation and obstruction theory follow from Olsson's log cotangent complex. Then we check that $\mathcal{K}_\Gamma(X)$ is an Artin stack locally of finite presentation using Artin's criteria.
2. *Boundedness.* Denote by $\mathcal{K}_{n,g}(\underline{X}, \beta)$ the stack of usual stable maps. Then there is a natural map $\mathcal{K}_\Gamma(X) \to \mathcal{K}_{n,g}(\underline{X}, \beta)$ by removing all log structure on log stable maps. Our proof is given by stratifying the stack $\mathcal{K}_{n,g}(\underline{X}, \beta)$, and constructing all possible log stable maps over each stratum.
3. *Properness.* We checked the valuative criterion for the natural map $\mathcal{K}_\Gamma(X) \to \mathcal{K}_{n,g}(\underline{X}, \beta)$. In fact the separatedness follows essentialy from the uniqueness of the minimal structure as one can tell from theorem 5.

6 A decomposition of $\mathcal{K}_\Gamma(X)$

Note that the discrete datum Γ induce discrete datum Γ_i with same markings, genus, and curve class, but only tangency condition with respect to D_i. Denote by X_i the log scheme $(\underline{X}, \mathcal{M}_i)$ with $\mathcal{M}_i$ given by the log structure associated to D_i. By the universal property of minimality, our stack of log stable maps has the following lovely property:

Theorem 8. *We have the following decomposition:*

$$\mathcal{K}_\Gamma(X) \cong \mathcal{K}_{\Gamma_1}(X_1) \times_{\mathfrak{M}_{n,g}} \cdots \times_{\mathfrak{M}_{n,g}} \mathcal{K}_{\Gamma_k}(X_k),$$

where we view $\mathcal{K}_\Gamma(X)$ and $\mathcal{K}_{\Gamma_i}(X_i)$ as log stacks with their universal log structure, and $\mathfrak{M}_{n,g}$ is the stack of usual pre-stable curves with the canonical log strcuture. The fibered product are taking over the category of fs log schemes.