Graph Curves of High Degree and their Secant Varieties

by

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1 Introduction

One area of research in algebraic geometry attempts to relate the algebraic properties of the syzygies of an ideal to the geometric properties of the variety determined by that ideal. Much is known about the geometric content of the syzygies of smooth curves of high degree, and our research focuses on a similar case: that of graph curves of high degree. It is our aim to describe the syzygies of these graph curves as specifically as possible, as well as study the syzygies of their secant varieties.

2 Graded Betti Diagrams

Our discussion of syzygies centers around minimal free resolutions and their corresponding Betti diagrams. Specifically, let $\mathbb{K}$ be a field, let M be a finitely-generated module over the polynomial ring $S = \mathbb{K}[x_0, \ldots, x_n]$, and suppose that the minimal free resolution of M is given by

$$0 \to F_r \to F_{r-1} \to \cdots \to F_1 \to F_0 \to M$$

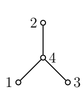
where $F_i = \oplus_j S(-j)^{\beta_{i,j}}$. Then, the *Betti diagram* of the resolution (or, of M) is the table

	0	1	$\cdots$	r
0	$\beta_{0,0}$	$\beta_{1,1}$	$\cdots$	$\beta_{r,r}$
1	$\beta_{0,1}$	$\beta_{1,2}$	$\cdots$	$\beta_{r,r+1}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
k	$\beta_{0,k}$	$\beta_{1,1+k}$	$\cdots$	$\beta_{r,r+k}$

An ideal is said to satisfy $N_{k,p}$ if it is generated by forms of degree k, and has only linear syzygies up until the pth column of its Betti diagram.

3 Graph Curves

A graph curve consists of several copies of $\mathbb{P}^1$ that are glued together as prescribed by a graph. For instance, the graph



gives a curve consisting of four lines, where lines 1, 2, and 3 intersect line 4, but do not intersect each other. A graph curve with d lines (nodes) and $d+g-1$ intersections (edges) has degree d and genus g.

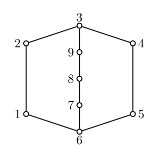
4 Syzygies of Graph Curves of High Degree

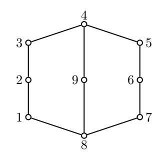
Motivated by the case of smooth curves, we confine our attention to graph curves *of high degree*, that is, with $d \geq 2g+1$. If we are able to extend results for smooth curves, then we will have:

Conjecture 1. *The homogeneous coordinate rings of graph curves of high degree have regularity 2 and are arithmetically Cohen-Macaulay. Thus, the Betti diagrams have three non-zero rows and $d-2$ non-zero columns.*

The Betti numbers themselves are more subtle, containing information about the specific structure of the graph curves.

Example 2. Consider, for instance, the two graph curves associated to the graphs below, each with $g = 2$ and $d = 9$.





When we embed these curves in $\mathbb{P}^7$, we get Betti diagrams:

	0	1	2	3	4	5	6
0	1	.	.	.	.	.	.
1	.	19	58	75	44	6	.
2	.	.	.	.	1	6	2

	0	1	2	3	4	5	6
0	1	.	.	.	.	.	.
1	.	19	58	75	44	7	.
2	.	.	.	.	2	6	2

This phenomenon – two Betti diagrams for two curves of the same degree and genus having the same shape but different entries – is interesting: we do not know if it can even occur for smooth curves of high degree. (It is, at least, not possible for $g \leq 3$.)

For comparison, the Betti diagram of a *smooth* curve of the same degree and genus is:

	0	1	2	3	4	5	6
0	1	.	.	.	.	.	.
1	.	19	58	75	44	5	.
2	.	.	.	.	.	6	2

So, the graph curves have quadratic syzygies "earlier" that the corresponding smooth curves. This is always true, and for a geometric reason.

Theorem 3. *Suppose X is a graph curve with genus g and degree $d \geq 2g+1$, and suppose that the corresponding graph has girth r. Then, $\beta_{r-2,r}$ is non-zero. In other words, X does not satisfy $N_{2,r-2}$.*

These "early" quadratic syzygies appear because there is a linear subspace intersecting X in "too many" points: a cycle of r lines spans a $\mathbb{P}^{r-1}$, and any $(r-2)$-dimensional hyperplane in this $\mathbb{P}^{r-1}$ therefore intersects X in r points. By a generalization of a result by Eisenbud, Green, Hulek, and Popescu, we have that X cannot satisfy $N_{2,r-2}$.

In the example, we see that the specific value of $\beta_{r-2,r}$ can vary, but that it does seem to be the first place that quadratic sysyzgies appear. We make the following:

Conjecture 4. *Let X be a graph curve of genus g, degree $d \geq 2g+1$, and girth r. Then, X satisfies $N_{2,r-3}$, and, up until the quadratic syzygies appear, the Betti numbers of X are given by $\beta_{k,k+1} = k\binom{d-g}{k+1} - g\binom{d-g-1}{k-1}$.*

Moreover, $\beta_{r-2,r}$ is the number of cycles of lines in X with length r.

We have verified this for $g = 2$ and $d = 5, \ldots, 16$, as well as $g = 3$ and $d = 10$.

5 Secant Varieties

If we require that $d \geq 2g+3$, then, again motivated by the smooth case, we expect the syzygies of the secant varieties of our graph curves to have nice properties. The secant varieties of the two graph curves to the left have Betti diagrams:

	0	1	2	3	4
0	1	.	.	.	.
1	.	.	.	.	.
2	.	12	16	.	.
3	.	.	.	5	.
4	.	.	1	4	3

	0	1	2	3	4
0	1	.	.	.	.
1	.	.	.	.	.
2	.	12	16	.	.
3	.	.	.	6	.
4	.	.	2	4	3

The "early" syzygies remain, again because of hyperplanes intersecting the variety in "too many" places. Less is known about the secant varieties of smooth curves, and so we limit ourselves to the following:

Conjecture 5. *Let X be a graph curve with genus g and degree $d \geq 2g+3$, and let Σ be its secant variety. Then, the homogeneous coordinate ring of Σ has regularity 4 and is arithmetically Cohen-Macaulay. If X has girth r, then Σ satisfies $N_{3,r-5}$, but not $N_{3,r-4}$. Until these non-linear syzygies appear, the Betti numbers of Σ are equal to those of the corresponding smooth variety.*