The relative tropical lifting problem

by

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1 Introduction

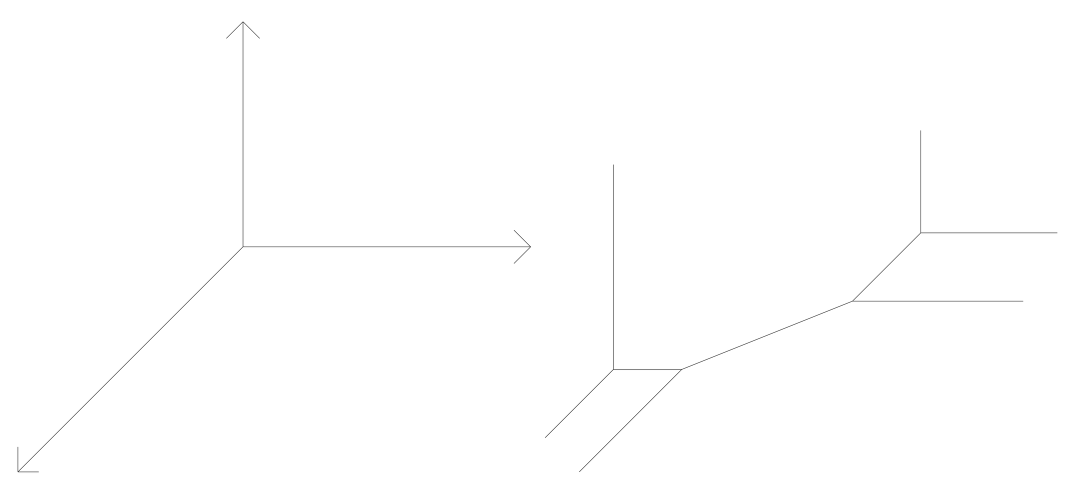
Tropical geometry is an emerging approach to algebraic geometry using combinatorial and discrete geometric techniques. For example, algebraic curves are modelled by certain types of graphs and surfaces by certain complexes of polygons. These discrete objects are known as **tropical varieties**. There is a **tropicalization map** that takes an ordinary variety embedded in an algebraic torus to a tropical variety.

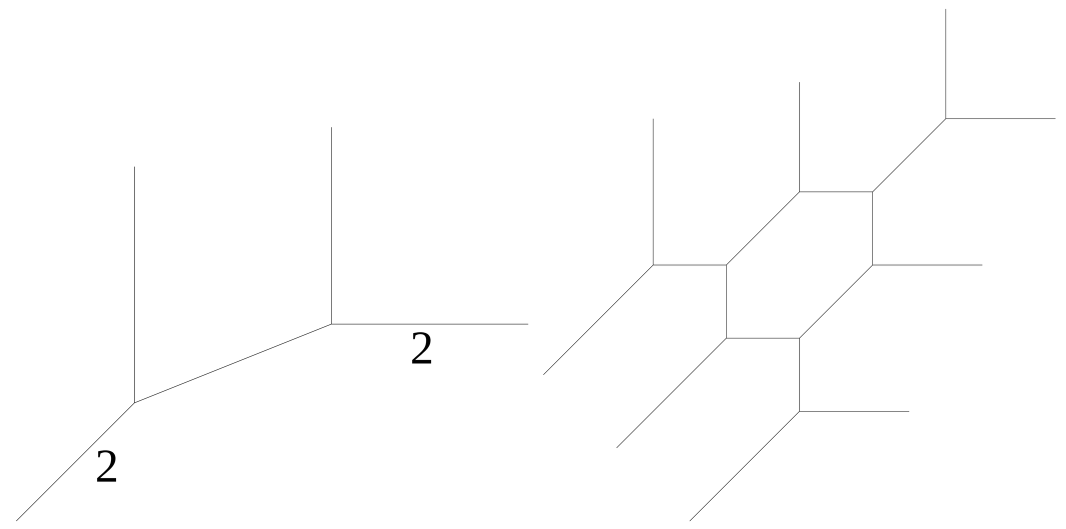
The general **tropical lifting problem** asks: which tropical varieties arise as tropicalizations? That is, which can be "lifted" to ordinary algebraic varieties? There are many special cases and variants of this problem. One such variant is the **relative** version: if one tropical variety contained in another and both can be lifted, can they be lifted to ordinary varieties with the same containment property?,

Both the tropical lifting problem and the relative version are very difficult in general, so many people have focused on special cases and on obstructions. In forthcoming work with Eric Katz, we produce a local obstruction to relative tropical lifting.

2 Tropical Varieties

A **polyhedral complex** in $\mathbb{R}^n$ is a collection of polyhedra that is closed under intersection. A **tropical variety of dimension** d is a polyhedral complex whose maximal elements are all d-dimensional, together with a positive integer weight $u(P)$ on each maximal element P such that a **balancing** or **zero-tension** condition holds. For curves, this condition is that the primitive edge vectors at each vertex, each multiplied by the corresponding edge weight, sum to zero.





These examples are, in order, a tropical line, two tropical conics, and a tropical cubic in $\mathbb{R}^2$. All edge weights are one except the two that are marked as 2 in the second conic.

Moving these curves around, notice that Bezout's theorem holds for tropical curves: for example, a line and a cubic intersect in three points, properly counted. This is an instance of how tropical geometry reflects ordinary algebraic geometry.

3 Tropicalizations

If V is an ordinary variety of pure dimension d in $\mathbb{C}^n$, let I be its defining ideal. For $\omega \in \mathbb{R}^n$, the **initial ideal** $\text{in}_\omega(I)$ is the ideal generated by the initial terms, with respect to ω of each polynomial in I. The **tropicalization** of V is the set

$$\mathcal{T}(V) = \{\omega \in \mathbb{R}^n : \text{in}_\omega(I)\text{does not contain a monomial.}\}$$

This set is a tropical variety.

More generally, replace $\mathbb{C}$ by a field with valuation such as the field of Puiseux series $\mathbb{C}\{\{t\}\}$ and modify the notion of initial ideal accordingly. This is necessary to produce tropical varieties whose maximal faces are not all unbounded, such as the conics and cubic above.

4 Lifting and Relative Lifting

Tropical Lifting Problem: Given a tropical variety $T \subseteq \mathbb{R}^n$, is there a variety X defined over $\mathbb{C}\{\{t\}\}$ such that $\mathcal{T}(X) = T$?

Relative Tropical Lifting Problem: Given tropical varieties $T \subseteq U$, each of which *can* be lifted, are there ordinary varieties $X \subseteq Y$ such that $\mathcal{T}(X) = T$ and $\mathcal{T}(Y) = U$?

In general, these are difficult problems, so consider some special cases:

1. Determining whether T lifts to a *linear space* depends [Ardila-Klivans] on the subtle question [Vámos] of matroid realizability. A famous example is the **Fano matroid**, realizable only over a field of characteristic 2.
2. A balanced tree in $\mathbb{R}^n$ always lifts to a rational curve. A balanced graph in $\mathbb{R}^n$ with a single cycle lifts to a curve of genus one if (not only if) the cycle is not contained in a hyperplane [Speyer].
3. There is a family of realizable tropical cubic surfaces in $\mathbb{R}^3$, each containing an infinite family of tropical lines Each line is also realizable by Speyer's result. Yet the relative lifting property **fails** [Vigeland]: the ordinary cubic surfaces obtained by lifting the tropical ones contain only the expected 27 lines.

5 An obstruction to relative lifting

Suppose a tropical curve T is contained in a tropical hypersurface U in $\mathbb{R}^n$, $n \geq 3$. Suppose a trivalent vertex v of T lies on a smooth vertex (i.e. the dual to a unimodular simplex) of U. (This situation is not surprising if only a finite number of curves of a certain type are to exist on the hypersurfaces, as in enumerative problems. If many vertices of T coincide with vertices of U, then T is more likely to be rigid.)

If T and U are to satisfy the relative lifting property in this scenario, then we can produce strong restrictions on the possible edge directions of T adjacent to v.

In the cases $n = 3$ and $n = 4$, we are able to enumerate all possible directions for edges of T that are adjacent to v. Vigeland's lines possess a vertex failing our criterion, so we can explain the failure of tropical lifting in this case.

6 References

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