The rank of a hypergeometric system

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1 Introduction

An **A-hypergeometric system** $M_A(\beta)$ of PDEs is determined by

an integer matrix A and a parameter vector β.

The solutions of $M_A(\beta)$ occur naturally in mathematics and physics. For example, within algebraic geometry, these functions occur in the study of roots of polynomials, toric residues, and Picard–Fuchs equations for the variation of Hodge structure of Calabi–Yau toric hypersurfaces.

Its **rank** is the dimension of its solution space, $\operatorname{Sol} M_A(\beta)$.

> **Main Question:** For a fixed matrix A, how does the rank of the hypergeometric system $M_A(\beta)$ vary with β?

These systems fit nicely into the context of toric geometry; they are the D-module equivalent to a toric ideal. We apply homological techniques to obtain geometric and combinatorial answers to this question.

2 A-hypergeometric systems

Let $A \in \mathbf{Z}^{d\times n}$. Assume that $\mathbf{Z}A = \mathbf{Z}^d$ and $\mathbf{N}A$ is pointed.

- **Weyl algebra**: $D = \mathbf{C}[x_1,\dots,x_n]\langle\partial_1,\dots,\partial_n\rangle \qquad (\partial_i x_i = x_i\partial_i + 1)$
- **Toric ideal**: $I_A = \langle \partial^u - \partial^v \mid u,v \in \mathbf{N}^n,\ Au = Av\rangle \subseteq \mathbf{C}[\partial]$

$$S_A = \mathbf{C}[\partial]/I_A \cong \mathbf{C}[\mathbf{N}A]$$

- **Euler operators**: $E_i = \sum_{j=1}^n a_{ij}x_j\partial_j \qquad (1 \le i \le d)$
- **Parameter**: $\beta \in \mathbf{C}^d \quad \rightsquigarrow \quad E-\beta = \{E_i - \beta_i\}_{i=1}^d$

> The **A-hypergeometric system** of A at β is
>
> $$M_A(\beta) := D/D\cdot\langle I_A, E-\beta\rangle = \mathcal{H}_0(S_A,\beta).$$

We view $M_A(\beta)$ as the 0th Koszul homology of $E-\beta$ on $D/D\cdot I_A$.

Let $\operatorname{Sol} M_A(\beta)$ denote the germs of holomorphic solutions on $M_A(\beta)$ at a generic point in $\mathbf{C}^n$.

3 The quadratic formula

For $A = \begin{bmatrix}1&1&1\\0&1&2\end{bmatrix}$, the affine toric algebra is $S_A = \dfrac{\mathbf{C}[\partial_1,\partial_2,\partial_3]}{I_A} = \dfrac{\mathbf{C}[\partial]}{\langle\partial_1\partial_3 - \partial_2^2\rangle}$.

With $\beta = \begin{bmatrix}0\\-1\end{bmatrix}$, the A-hypergeometric system is

$$M_A(\beta) = D/D\cdot\langle\partial_1\partial_3 - \partial_2^2, x_1\partial_1 + x_2\partial_2 + x_3\partial_3,\ x_2\partial_2 + 2x_3\partial_3 + 1\rangle.$$

The solutions to $x_3T^2 + x_2T^1 + x_1T^0 = 0$,

$$\frac{-x_2 \pm \sqrt{x_2^2 - 4x_1x_3}}{2x_3},$$

span $\operatorname{Sol}(M_A(\beta))$. (This is true more generally!)

4 Rank jumps and the exceptional arrangement

Let $\operatorname{vol}(A)$ denote $d!$ times the Euclidean volume of the convex hull of the origin and the columns of A. By work of Gelfand–Graev–Kapranov–Zelevinsky, Adolphson, Cattani–D'Andrea–Dickenstein, Saito–Sturmfels–Takayama, Matusevich–Miller–Walther,

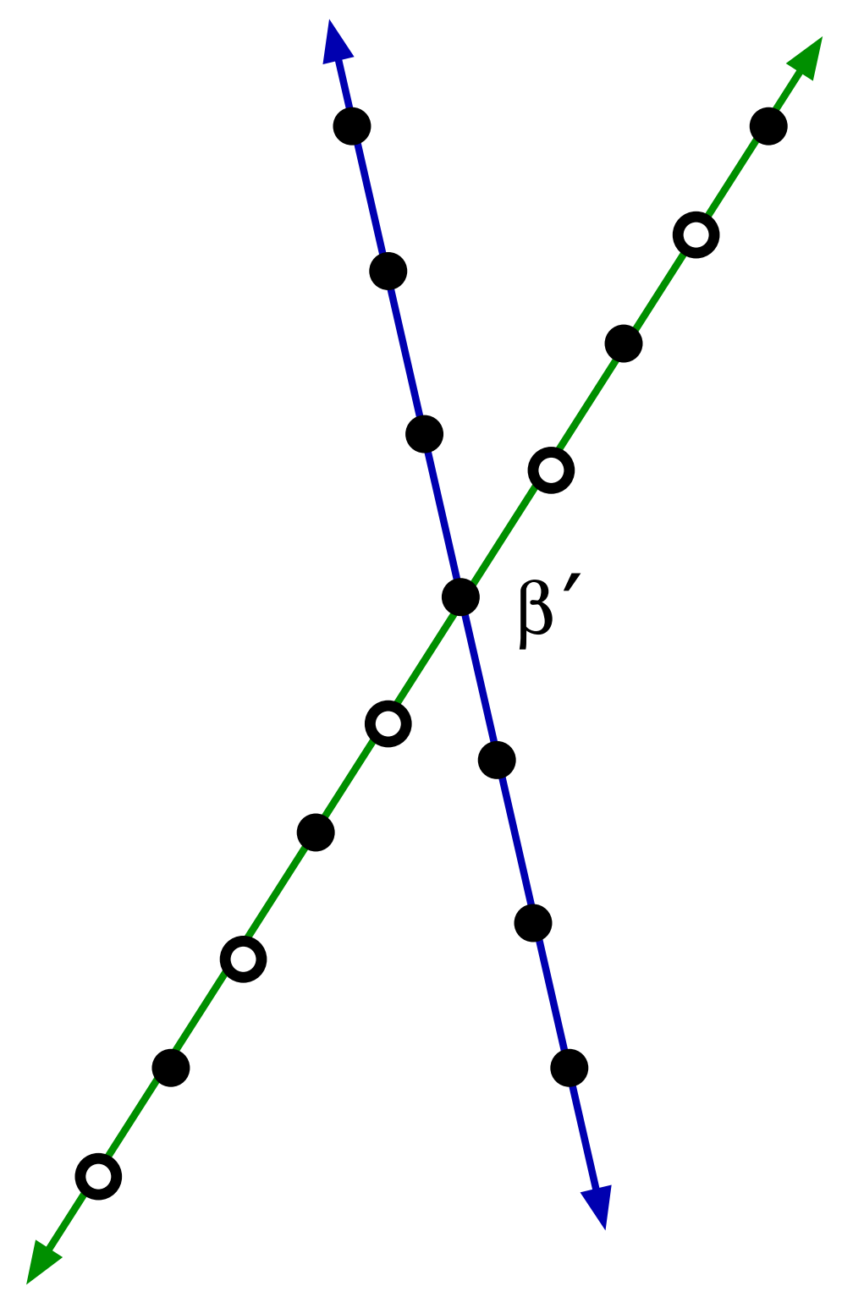
$$\operatorname{vol}(A) \le \operatorname{rank} M_A(\beta) < \infty \quad \forall A,\beta.$$

For generic $\beta \in \mathbf{C}^d$, $\operatorname{vol}(A) = \operatorname{rank} M_A(\beta)$, but there is not equality in general.

- **Rank jump** of A at β: $j(\beta) = \operatorname{rank} M_A(\beta) - \operatorname{vol}(A)$
- **Exceptional arrangement**: $\mathcal{E}_A = \{\beta \in \mathbf{C}^d \mid j(\beta) > 0\}$

> **Theorem** [Matusevich–Miller–Walther]: With $A = [a_1\ a_2\ \cdots\ a_n]$ and $\deg(\partial_i) = a_i$,
>
> $$\mathcal{E}_A = \overline{-\deg\left[\oplus_{i=0}^{d-1}\operatorname{Ext}^{n-i}_{\mathbf{C}[\partial]}(S_A,\mathbf{C}[\partial])\left(-\textstyle\sum_{j=1}^n a_j\right)\right]}^{\text{Zar}}$$
>
> and $\operatorname{rank} M_A(-)$ induces a *rank stratification* on $\mathbf{C}^d$.

5 The combinatorics of rank jumps

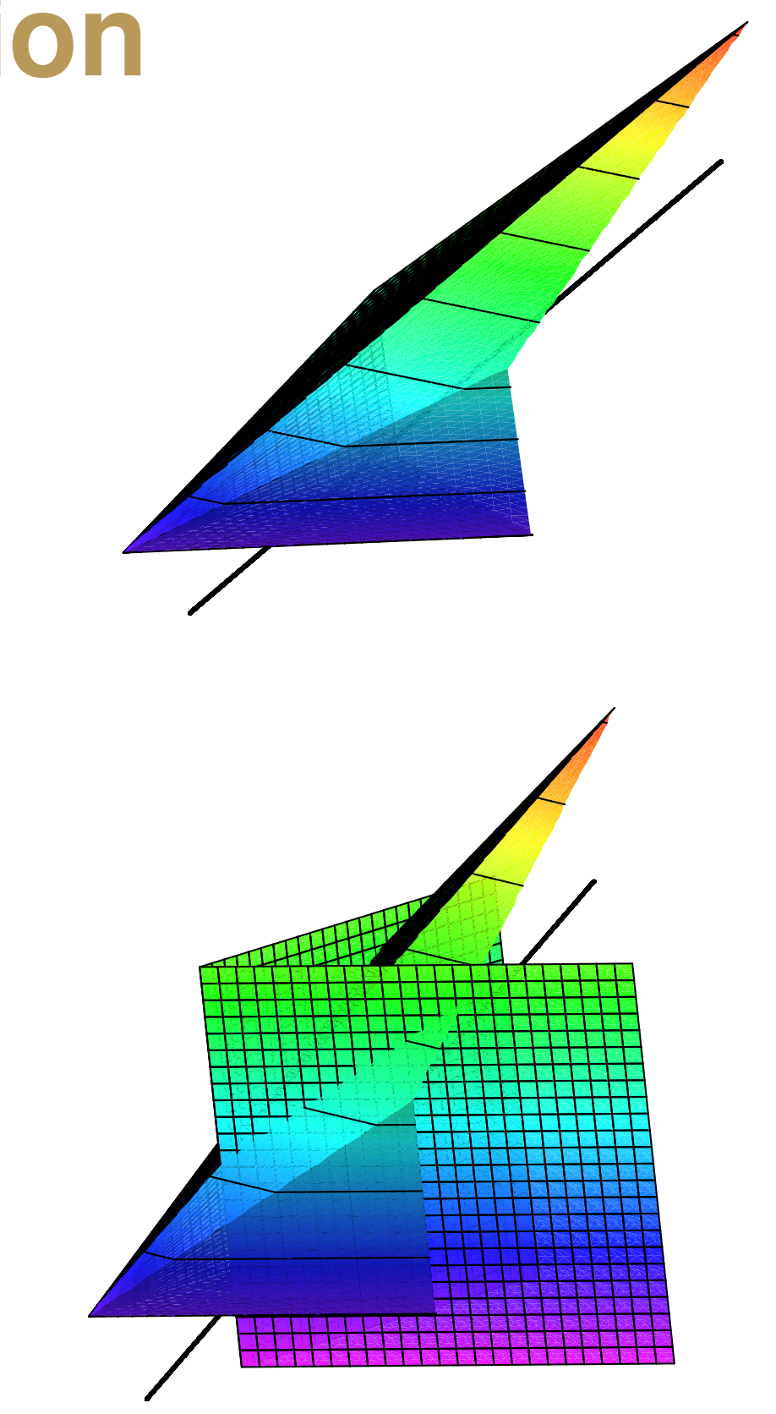


- $A = \begin{bmatrix}2&1&1&1&1&1&1\\0&1&1&1&1&0&1\\0&0&1&3&4&0&2\\0&0&0&0&0&1&1\end{bmatrix}$, $\beta' = \begin{bmatrix}1\\\frac{1}{2}\\0\end{bmatrix}$, with $\underbrace{}_{F_1}$ under the first columns and $\underbrace{}_{F_2}$ under the last columns
- $\operatorname{vol}(F_i) = 1,\ \operatorname{codim}(F_i) = 3$
- $\mathcal{E}_A = L_1 \cup L_2$
- $\beta \neq \beta'$:
  - ○ $\beta \in L_2:\quad j(\beta) = [3-1]\cdot 1 = 2$
  - ○ $\beta \in L_1:\quad j(\beta) = 2\cdot 2 = 4$
- $j(\beta') = 4 \neq 2+4$

6 A surprising rank stratification

- $A = \begin{bmatrix}2&3&2&2&0&0&0&0&2&5&3\\0&0&0&0&2&3&2&2&2&3&5\\0&0&1&2&0&0&1&2&5&7&7\end{bmatrix}$, $\beta' = \begin{bmatrix}1\\1\\0\end{bmatrix}$, with braces F_1, F_2, F_3 under the column groups $\underbrace{2\,3\,2\,2}_{F_1}$, $\underbrace{0\,0\,0\,0}_{F_2}$, $\underbrace{2\,5}_{F_3}$
- $\mathcal{E}_A = [\beta' + \mathbf{C}F_3] \cup \{6 \text{ points}\}$
- $j_A(\beta) = \begin{cases}1 & \text{if } \beta \in [\beta' + \mathbf{C}F_3] \setminus \beta'\\ 2 & \text{if } \beta = \beta'\end{cases}$
- $\beta \in \mathbf{R}_{\ge 0}A \cap [\beta' + \mathbf{C}F_3] \setminus \beta'$:

$$\mathbf{E}^\beta = \beta' + \mathbf{Z}F_3$$
$$\mathbf{E}^{\beta'} = \cup_{i=1}^3\,[\beta' + \mathbf{Z}F_i]$$



7 A homological approach

Recall that $M_A(\beta) = \mathcal{H}_0(S_A,\beta)$.

- $\mathcal{H}_\bullet(-,\beta)$: Koszul homology of $E-\beta$ on $D\otimes_{\mathbf{C}[\partial]} -$
- $0 \longrightarrow S_A \longrightarrow \mathbf{C}[\mathbf{Z}A] \longrightarrow Q \longrightarrow 0$
- **Ranking lattices** at β: $\mathbf{E}^\beta = \bigcup(b + \mathbf{Z}F) \subseteq \deg(Q)$,

where $\beta \in (b + \mathbf{C}F)$ and $(b + \mathbf{Z}F) \cap (\mathbf{N}A + \mathbf{Z}F) = \varnothing$.

> **Theorem** [–]: The rank jump $j_A(\beta)$ of $M_A(\beta)$ can be computed from the combinatorics of the ranking lattices $\mathbf{E}^\beta$.

8 The ranking lattice stratification

There are natural stratifications given by flats of the following arrangements:

- $\overline{\deg(Q)}^{\text{Zar}}$ and
- $\overline{-\deg\left[\operatorname{Ext}^{n-i}_{\mathbf{C}[\partial]}(S_A,\mathbf{C}[\partial])\left(-\textstyle\sum_{j=1}^n a_j\right)\right]}^{\text{Zar}}$ for $0 \le i \le d-1$.

> **Theorem [–]:** The stratification of $\mathcal{E}_A$ by ranking lattices is the coarsest stratification of $\mathbf{C}^d$ refining all of these arrangement stratifications.

Stratifications of $\mathbf{C}^d$:

- $\mathcal{R}$: *by* $\operatorname{rank} M_A(\beta)$
- $\mathcal{E}$: *by arrangement flats*
- $\mathcal{L}$: *by ranking lattices* $\mathbf{E}^\beta$

Comparison of stratifications:

- $\mathcal{E}$ and $\mathcal{R}$ may not be comparable.
- $\mathcal{L}$ refines $\mathcal{E}$ and $\mathcal{R}$.
- $\mathcal{L}$ may be finer than $\mathcal{E}\cap\mathcal{R}$.

9 Next steps

Question: How does the solution space of $M_A(\beta)$ vary with β?

Curious Example: For $A = \begin{bmatrix}1&1&1&1\\0&1&3&4\end{bmatrix}$, $\operatorname{rank} M_A\left(\frac{1}{2}\right) = 5 = \operatorname{vol}(A) + 1$.

Below is a basis for the solution space $\operatorname{Sol} M_A(\beta)$:

$$\frac{x_2^2}{x_1},\ \frac{x_3^2}{x_4},\ \text{ and three infinite series.}$$

Interestingly, the following is a generator for $H^1_{\langle\partial\rangle}(S_A)$:

$$\left(\frac{\partial_2^2}{\partial_1}, \frac{\partial_3^2}{\partial_4}\right) \in S_A[\partial_1^{-1}] \oplus S_A[\partial_4^{-1}]$$

There is hope that ranking lattices provide a link between an explicit basis of $\operatorname{Sol} M_A(\beta)$ and the local cohomology modules $H^\bullet_{\langle\partial\rangle}(S_A)$ that would explain this fascinating phenomenon.