Degeneration of hyperkähler manifolds and Nagai's conjecture

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Definitions

A **compact hyperkähler (HK) manifold** is a simply connected compact Kähler manifold X whose $H^{2,0}(X)$ is generated by a single global holomorphic symplectic 2-form.

Let Δ be the complex unit disk and $\Delta^* = \Delta \setminus \{0\}$. A **degeneration** of X is a flat proper surjective morphism $\pi : \mathfrak{X} \to \Delta$ whose restriction $\pi_{|\Delta^*} : \mathfrak{X}^* \to \Delta^*$ is smooth and the special fiber $\pi^{-1}(t)$, $t \neq 0$ is isomorphic to X.

Consider the monodromy representation

$$\rho_{2k} : \pi_1(\Delta^*, t) = \mathbb{Z} \to \mathrm{GL}(H^{2k}(X, \mathbb{Q})).$$

We call $T_{2k} = \rho_{2k}(1)$ its $2k$-th monodromy operator, and $N_{2k} = \log T_{2k}$ its $2k$-th log monodromy operator. The largest nonnegative integer ν_{2k} with $(N_{2k})^{\nu_{2k}} \neq 0$ is called its **nilpotency index**.

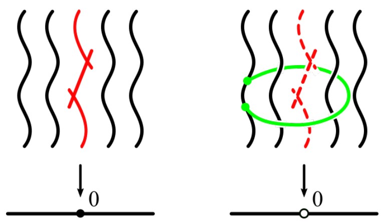

Figure 1: Degeneration of HK manifolds. The black fibers are HK manifolds deformation equivalent to X. The red fiber is the degenerated central fiber X_0. The green loop represents the monodromy transformation around 0.

Nagai's Conjecture

Let X be a HK manifold of dimension $2n$ and $\pi : \mathfrak{X} \to \Delta$ its degeneration. Consider its $2k$-th log monodromy operator $N_{2k} \in \mathrm{End}(H^{2k}(X, \mathbb{Q}))$ and let ν_{2k} be its nilpotency index. Then we have

$$\nu_{2k} = k\nu_2 \quad \text{for} \quad 0 \leq k \leq n.$$

Main Result

Nagai's conjecture holds for all currently known examples—$\mathrm{K3}^{[n]}$, Kum_n, OG6 and OG10—of HK manifolds, *under the assumption* that OG10 has no odd cohomology.

Hard Lefschetz for Hyperkähler Manifolds

The classical hard Lefschetz theorem for smooth projective varieties says:

$H^*(X, \mathbb{Q})$ admits an $\mathfrak{sl}(2, \mathbb{Q})$-module structure.

Verbitsky, Looijenga and Lunts discovered a stronger version of hard Lefschetz theorem for HK manifolds:

$H^*(X, \mathbb{Q})$ admits a $\mathfrak{g}$-module structure,

where $\mathfrak{g} = \mathfrak{so}\left(H^2(X, \mathbb{Q}) \oplus \mathbb{Q}^2,\ q_{\mathrm{BB}} \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\right)$.

This imposes surprisingly rich symmetries on the cohomology of HK manifolds. Representation theory tells us this can be conveniently modeled in terms of Young diagrams $\lambda = (\lambda_1, \cdots, \lambda_r)$.

Let λ be a Young diagram and V_λ the irreducible $\mathfrak{g}$-module associated to λ. Then we can describe the hard Lefschetz decomposition as

$$H^*_{\mathrm{even}}(X, \mathbb{Q}) = \bigoplus_\lambda V_\lambda^{\oplus m_\lambda}. \tag{1}$$

The Criterion

Nagai's conjecture holds if and only if

$$\lambda_1 + \lambda_2 + \lambda_3 \leq n \quad \text{for all Young diagrams } \lambda = (\lambda_1, \cdots, \lambda_r) \text{ appearing in (1).}$$

Sketch of Ideas

Step 1 Using Sullivan and Verbitsky's theorem on mapping class groups, Soldatenkov proved that the second monodromy already contains almost all information about the total monodromy. Their relations are encoded in the $\bar{\mathfrak{g}}$-module structure on $H^{2k}(X, \mathbb{Q})$, where $\bar{\mathfrak{g}} = \mathfrak{so}(H^2(X, \mathbb{Q}),\ q_{\mathrm{BB}})$.

Step 2 The Hodge structure $H^2(X, \mathbb{Q})$ is of K3 type, so we can compute its LMHS for for each type I/II/III of the degeneration. This enables us to compute the second monodromy explicitly, and hence all the higher monodromy completely.

Step 3 We need to "lift up" the representation theory on each $H^{2k}(X, \mathbb{Q})$ to the entire $H^*_{\mathrm{even}}(X, \mathbb{Q})$. This is done by comparing the weight lattices of $\bar{\mathfrak{g}}$ and $\mathfrak{g}$, and hence their representation theory. This proves the main criterion.

Step 4 Show all currently known examples of HK manifolds satisfy the criterion. For $\mathrm{K3}^{[n]}$ and Kum_n types, one need to use their Hodge structures and Mumford-Tate groups. The cohomology of OG10 is less understood, so we imposed a plausible condition and proved the criterion under that condition.

History & Motivations

In 1977, Kulikov classified degenerations of K3 surfaces, up to finite étale base change and birational modifications. There are three types I/II/III of them, captured by the nilpotency index $\nu_2 = 0, 1, 2$ of the monodromy on the second cohomology.

Compact hyperkähler manifolds are the higher dimensional analogue of K3 surfaces. Regardless of their dimension, their information is encoded in the *second cohomology* $H^2(X, \mathbb{Z})$. Thus, one can study the monodromy on H^2 and imitate the Kulikov classification to study degenerations of HKs. But what happens to the higher cohomologies?

In 2008, Nagai discovered that the higher nilpotency index ν_{2k} becomes precisely the k times of ν_2 for some degenerations of HKs, hinting the fact that monodromy of H^2 governs the behavior of monodromy of H^{2k}. He conjectured this would be true for any degeneration of HKs.

Recently, Kollár-Laza-Saccà-Voisin established a partial generalization of Kulikov classification for degeneration of HKs. In particular, they proved Nagai's conjecture is indeed true for Type I and III degenerations.

Here we give a systematic examination for the underlying reason of Nagai's conjecture, and prove it for most cases of currently known HKs.

Example: $\mathrm{K3}^{[4]}$ type

Let X be a $\mathrm{K3}^{[4]}$ type HK manifold. Its hard Lefschetz decomposition is

$$H^*_{\mathrm{even}}(X, \mathbb{Q}) = V_{\ydiagram{4}} \oplus V_{\ydiagram{2,1}} \oplus V_{\ydiagram{2}} \oplus V_\varnothing,$$

All Young diagrams above satisfy the criterion, so Nagai's conjecture holds for any degeneration of $\mathrm{K3}^{[4]}$ type HK manifolds.

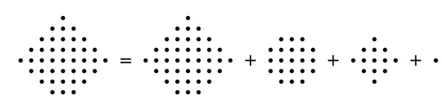

Figure 2: Hard Lefschetz decomposition of the cohomology of $\mathrm{K3}^{[4]}$, visualized in terms of the Hodge diamond.

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