# Seshadri constants for vector bundles

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$X$ denotes a projective variety over a field $k = \bar{k}$.

## Seshadri constants for line bundles (Demailly 1992)

Let $L$ be a nef line bundle on $X$. Letting $\pi\colon \widetilde{X} \to X$ be the blowup of $X$ at $x \in X$ with exceptional divisor $E$, the Seshadri constant of $L$ at $x$ is

$$\varepsilon(L; x) := \sup\{t \in \mathbf{R}_{\geq 0} \mid \pi^*L(-tE) \text{ is nef}\}.$$

This measures the local positivity of $L$ at $x$.

Seshadri constants are interesting because they are related to Fujita's conjecture and capture subtle geometric properties of both $X$ and $L$; see Lazarsfeld (2004), Ch. 5.

Seshadri constants for line bundles were introduced by Demailly (1992), and Seshadri constants for vector bundles first appeared in Beltrametti–Schneider–Sommese (1993, 1996) and Hacon (2000).

We define a more general version:

## Definition: Seshadri constants for vector bundles

Let $\mathcal{V}$ be a vector bundle on $X$. Letting $\pi\colon \widetilde{X} \to X$ be the blowup of $X$ at $x \in X$ with exceptional divisor $E$, the Seshadri constant of $\mathcal{V}$ at $x$ is (roughly)

$$\varepsilon(\mathcal{V}; x) := \sup\left\{ t \in \mathbf{R}_{\geq 0} \;\middle|\; \begin{array}{c} \pi^*\mathcal{V}\langle -tE\rangle \text{ is nef on} \\ \text{strict transforms of} \\ \text{curves through } x \end{array} \right\}$$

Here, $\mathcal{W}\langle - \rangle$ denotes the formal twist of a vector bundle by an $\mathbf{R}$-divisor.

Our Seshadri constants satisfy many of the usual properties of Seshadri constants:

1. A Seshadri-type ampleness criterion:
$$\mathcal{V} \text{ is ample} \iff \inf_{x \in X} \varepsilon(\mathcal{V}; x) > 0.$$
2. $\varepsilon(S^m\mathcal{V}; x) = m \cdot \varepsilon(\mathcal{V}; x) = \varepsilon(\mathcal{V}^{\otimes m}; x)$.
3. A description via jet separation for ample $\mathcal{V}$.
4. Semicontinuity.
5. $\mathbf{B}_+(\mathcal{V}) = \{x \in X \mid \varepsilon(\mathcal{V}; x) = 0\}$ for nef $\mathcal{V}$.
6. Lower bounds imply jet separation for adjoint bundles for big and nef $\mathcal{V}$ (over $\mathbf{C}$).

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Our first application identifies new nef classes on self-products of curves. Recall the following:

## Conjecture (see Lazarsfeld (2004), Remark 1.5.10)

Let $C$ be a very general smooth projective curve of genus $g \gg 0$ over $\mathbf{C}$. Then,

$$(\sqrt{g} + 1)(f_1 + f_2) - \delta \in \mathrm{Nef}^1(C \times C),$$

where $f_1$ and $f_2$ (resp. $\delta$) are the classes of the fibers of the projections $p_1$ and $p_2$ (resp. the class of the diagonal in $C \times C$).

In the spirit of this conjecture, we identify new nef classes on $C \times C$ in the following result.

## Theorem A: New nef classes on $C \times C$ (Fulger and M— 2019)

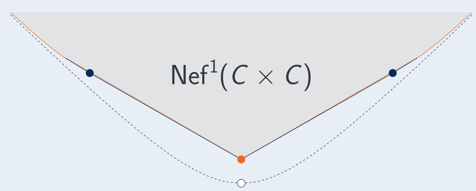
Let $C$ be a general smooth projective curve of genus $g \geq 3$ over $\mathbf{C}$. Then, we have

$$df_1 + \left(1 + \frac{g}{d - g}\right) f_2 - \delta \in \mathrm{Nef}^1(C \times C)$$

for every integer $d \geq \lfloor 3g/2 \rfloor + 3$, where $f_1$, $f_2$, and $\delta$ are as above.

This improves known results due to Rabindranath (2019) (following Vojta (1989)) and Kouvidakis (1993) in the range $\lfloor 3g/2 \rfloor + 3 \leq d < 2g$. This range is non-empty when $g \geq 7$.

## The nef cone $\mathrm{Nef}^1(C \times C)$ when $g = 7$




Our new nef classes are above the blue line segments. The orange dot in the middle comes from Kouvidakis (1993), and the curved portions on the sides are from Rabindranath (2019). The white dot is the class predicted by the conjecture above.

The proof of Theorem A uses $\varepsilon(\mathcal{V}; x)$ to show that the nefness of classes of the form above is equivalent to the asymptotic semi-stability of certain Lazarsfeld–Mukai-type bundles

$$R^{n-1}(nL) := p_{2*}\bigl(p_1^*\mathcal{O}_C(nL) \otimes \mathcal{O}_{C\times C}(-n\Delta)\bigr),$$

where $\deg L = d$. The generality assumption is not needed if $d \geq 2g + 2$.

Our second application is in the direction of a relative Fujita-type conjecture due to Popa and Schnell (2014), and generalizes a theorem of Dutta and M— (2019).

## Theorem B: Effective jet separation of direct images (Fulger and M— 2019)

Let $f\colon Y \to X$ be a surjective morphism of projective varieties over $\mathbf{C}$. Let $(Y, \Delta)$ be an lc pair and let $L$ be a big and nef line bundle on $X$. If $m(K_Y + \Delta)$ is Cartier for some integer $m \geq 1$, then

$$f_*\mathcal{O}_Y\bigl(m(K_Y + \Delta)\bigr) \otimes L^{\otimes l}$$

separates $s$-jets at all general points $x \in X$ for all $l \geq m(n(n + s) + 1)$, where $\dim X = n$.

We actually prove a version for vector bundles which assumes a lower bound on $\varepsilon(\mathcal{V}; x)$.

Our last application characterizes $\mathbf{P}^n_k$. This is inspired by a theorem of Mori (1979), which can be restated as

$$X \simeq \mathbf{P}^n_k \iff \inf_{x \in X} \varepsilon(T_X; x) > 0.$$

## Theorem C: Characterizing $\mathbf{P}^n_k$ (Fulger and M— 2019)

Assume $X$ is smooth of dimension $n$. Suppose that $\varepsilon(T_X; x_0) > 0$ for some $x_0 \in X$, and that either

1. $X$ is Fano,
2. $\operatorname{char} k = 0$ and $x_0$ is general, or
3. $\dim X = 2$.

Then, $X \simeq \mathbf{P}^n_k$.

We conjecture that the extra assumptions (1)–(3) can be removed. The main ingredients are respectively:

1. Mori (1979),
2. Cho–Miyaoka–Shepherd-Barron (2002), and
3. the Enriques classification of minimal surfaces.

## Some computations of $\varepsilon(T_X; x)$

If $X$ is a homogeneous space, then

$$\varepsilon(T_X; x) = \begin{cases} 2 & \text{if } X \simeq \mathbf{P}^1_k; \\ 1 & \text{if } X \simeq \mathbf{P}^n_k, \text{ where } n \geq 2; \\ 0 & \text{otherwise.} \end{cases}$$

Theorem C was also inspired by similar results for $\varepsilon(\omega_X^{-1}; x_0)$ due to Bauer–Szemberg (2009), Liu–Zhuang (2018), M— (2018), and Zhuang (2017, 2018).

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