The group of regular birational maps is infinitely-generated

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Motivation via Finite Fields

Consider a birational map $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined over a finite field $\mathbb{F}_q$. Furthermore, assume that both f and f^{-1} have no $\mathbb{F}_q$-points in their indeterminacy loci. In down-to-earth terms, the rational map f is given by

$$f([x, y, z]) = [f_1(x, y, z) : f_2(x, y, z) : f_3(x, y, z)]$$

where f_1, f_2, f_3 are homogenous polynomials of the same degree. The condition that the indeterminacy locus $\mathrm{Ind}(f)$ has no $\mathbb{F}_q$-points in $\mathbb{P}^2$ amounts to saying that the polynomials f_1, f_2, f_3 have no common (nonzero) solution with coordinates in $\mathbb{F}_q$.

Observation. Such a map f gives rise to a bijection

$$f : \mathbb{P}^2(\mathbb{F}_q) \to \mathbb{P}^2(\mathbb{F}_q)$$

Question 1. Consider a bijection $\sigma : \mathbb{P}^2(\mathbb{F}_q) \to \mathbb{P}^2(\mathbb{F}_q)$ which can be interpreted as a permutation $\sigma \in S_{q^2+q+1}$. Does there exist a birational map $f : \mathbb{P}^2 \to \mathbb{P}^2$ which induces σ?

Question 2. The set of all such birational maps forms a group $\mathrm{BCr}_2(\mathbb{F}_q)$ under composition. Is this group finitely-generated?

Introduction

The birational automorphisms of $\mathbb{P}^2$ defined over a field k form the *Cremona group* $\mathrm{Cr}_2(k)$ under composition. Let $\mathrm{Ind}(f) \subset \mathbb{P}^2$ denote the set of points where $f \in \mathrm{Cr}_2(k)$ is not defined. We are interested in the subgroup

$$\mathrm{BCr}_2(k) := \{f \in \mathrm{Cr}_2(k) : \mathrm{Ind}(f)(k) = \mathrm{Ind}(f^{-1})(k) = \emptyset\}.$$

Elements of $\mathrm{BCr}_2(k)$ are called *regular birational maps*. In particular, when $k = \mathbb{F}_q$ is a finite field, every element in $\mathrm{BCr}_2(\mathbb{F}_q)$ induces a permutation of the finite set $\mathbb{P}^2(\mathbb{F}_q)$.

Examples. Elements of $\mathrm{PGL}_3(k)$ are examples of birational maps. Another example is the standard quadratic transformation $\tau : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by $\tau([x : y : z]) = [yz : zx : xy]$.

Theorem (Noether and Castelnuovo 1901)

Suppose k is algebraically closed. Then $\mathrm{Cr}_2(k)$ is generated by $\mathrm{PGL}_3(k)$ and the standard quadratic involution τ above.

Theorem (Cantat 2012)

Suppose k is an arbitrary field. Then Cremona group $\mathrm{Cr}_2(k)$ is infinitely-generated.

Theorem (A.-Lai-Nakahara-Zimmermann)

Suppose k is a perfect field admitting a quadratic extension of degree of 2. Then the group of regular birational maps $\mathrm{BCr}_2(k)$ is infinitely-generated. Furthermore, $\mathrm{BCr}_2(k)$ is non-normal subgroup of infinite index of $\mathrm{Cr}_2(k)$.

Idea behind infinite-generation

For each $f \in \mathrm{BCr}_2(k)$, define L_f to be the minimal field extension of k such that all the indeterminacy points of f and f^{-1} are defined over L_f. If $\mathrm{BCr}_2(k)$ were finitely-generated by $f_1, f_2, ..., f_r$ then for each $f \in \mathrm{BCr}_2(k)$, L_f would be contained in the compositum of $L_{f_1}, ..., L_{f_r}$, and in particular

$$[L_f : k] \leq [L_{f_1} : k] \cdot [L_{f_2} : k] \cdots [L_{f_r} : k]$$

which would imply that the set

$$\{[L_f : k] : f \in \mathrm{BCr}_2(k)\}$$

is bounded. Therefore, it is enough to prove that, $[L_f : k]$ achieves arbitrarily large values as f varies in $\mathrm{BCr}_2(k)$.

Constructing the Cremona map: Algebra

For each $n \geq 2$, pick $n+4$ points on the plane where the 4 points and n points separately form two Galois orbits. Consider the linear system of degree $2n+1$ curves passing through the 4 points with multiplicity n, and through the n points with multiplicity 2. See Figure 1 for the case $n = 6$. Assuming the points are in sufficiently general position, this vector space has dimension

$$\binom{2n+3}{2} - 4\binom{n(n+1)}{2} - 3n = 3$$

since each multiplicity k point imposes $k(k+1)/2$ linear conditions. Choose a basis $\{f_1, f_2, f_3\}$ for this vector space, and define a rational map $f : \mathbb{P}^2 \to \mathbb{P}^2$ such that

$$f(x, y, z) = [f_1(x, y, z) : f_2(x, y, z) : f_3(x, y, z)]$$

One can check that $f \in \mathrm{BCr}_2(k)$, and the indeterminacy locus consists of exactly $n+4$ points above. Moreover, $[L_f : k] \geq n$ and we seem to be done!

Subtlety: How can we choose the points so they are in general position *and* form Galois orbits?

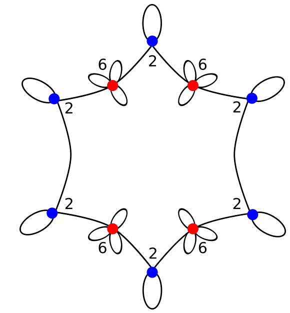

Figure 1: Degree 13 curve passing through 10 points ($n = 6$)

Constructing the Cremona map: Geometry

There is a geometric construction for the Cremona map discussed in the previous section. Here is a key diagram:

$$\begin{tikzcd}
 & & X \arrow[dl] \arrow[dr] & & \\
 & \mathcal{C} \arrow[dl] & & \mathcal{C}' \arrow[dr] & \\
\mathbb{P}^2 \arrow[rrrr, dashed, "f"] & & & & \mathbb{P}^2.
\end{tikzcd}$$

The spaces $\mathcal{C}$ and $\mathcal{C}'$ will be both conic bundles.

Let T be a quadratic extension of k. Pick four points $\{a_1, a_2, b_1, b_2\}$ in $\mathbb{P}^2(T)$ such that $\{a_1, a_2\}$ and $\{b_1, b_2\}$ form $\mathrm{Gal}(T/k)$-orbits, and no three of these four points are collinear.

Step 1. Blow-up $\mathbb{P}^2$ along $\{a_1, a_2, b_1, b_2\}$ to obtain a conic bundle $\mathcal{C} \to \mathbb{P}^1$, which is fibered in the conics passing through $\{a_1, a_2, b_1, b_2\}$. Let $\ell_1 \subset \mathbb{P}^2$ be a line over T passing through a_1, but not a_2, b_1, b_2, and let ℓ_2 be its $\mathrm{Gal}(T/k)$-conjugate.

Step 2. There exists a closed point $x \in \ell_1$ defined over an even degree extension K/k but not over any proper subfield, such that r of the $\mathrm{Gal}(K/k)$-conjugates of x lie on ℓ_1 (resp. ℓ_2), where $r = \frac{1}{2}[K : k]$ and such that all $n = 2r$ conjugates of x lie in the distinct fibers of $\mathcal{C} \to \mathbb{P}^1$.

Step 3. Blow-up $\mathcal{C}$ at these n points to get a map $X \to \mathcal{C}$.

Step 4. Blow-down the strict transforms of the n blown-up points to get $\mathcal{C}'$. Check that the resulting surface $\mathcal{C}'$ is smooth and admits a conic fibration. Moreover, $\mathcal{C}'$ is also a blow-up of $\mathbb{P}^2$ at four points.

References

[Can09] Serge Cantat, *Birational permutations*, C. R. Math. Acad. Sci. Paris **347** (2009), no. 21-22, 1289–1294. MR 2561040

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