Genus six curves, K3 surfaces, and stable pairs

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Background and motivation

Theorem(Artebani-Kondo [1]). There is a birational period map

$$\varphi : \mathcal{M}_6 \dashrightarrow D/\Gamma.$$

The target of φ is a period domain parametrizing the Hodge structures of certain $K3$ surfaces. To describe the construction of φ, we need the following two facts:

Fact 1. The canonical model of a general curve C of genus six is a quadric section of smooth quintic del Pezzo surface Σ_5 embedded anti-canonically in $\mathbb{P}^5$. In particular, $C \sim -2K_{\Sigma_5}$.

Fact 2. Let C as in Fact 1. If

$$\pi : X_C \to \Sigma_5$$

is a double cover of Σ_5 branched along C, then X_C is a $K3$ surface.

The construction of φ is as follows:

$$\varphi : [C] \mapsto \text{period point of } X_C.$$

Motivating question. How can we resolve φ in a modular way?

In other words, we want a diagram

$$\begin{array}{ccc} & \mathfrak{U} & \\ \swarrow & & \searrow \\ \overline{\mathcal{M}}_6 & \overset{\varphi}{\dashrightarrow} & (D/\Gamma) \end{array}$$

where we have a modular interpretation of $\mathfrak{U}$.

Main idea. Since φ depends on the existence of a surface containing a given curve, the space $\mathfrak{U}$ should parametrize surface-curve pairs (S, C) with $C \subset S$.

We note that the construction of φ does not extend to "special" curves.

Definition. A smooth curve of genus six is *special* if it is hyperelliptic, trigonal, bielliptic, or isomorphic to a smooth plane quintic curve.

It is natural to ask: Do moduli spaces of surface-curve pairs exist? Spaces of *stable* pairs do. These spaces are of intrinsic interest and are well studied. The existence of the well-known KSBA compactification of spaces of stable pairs is due to Kollár, Shepherd-Barron, and Alexeev. This construction was modified by Hacking ([2]) and later generalized by Deopurkar and Han ([3]).

Definition(Deopurkar-Han [3]). Fix relatively prime positive integers m and n with $m \leq n$. A *stable pair of type* (m, n) is a pair (S, C) where S is a projective surface and C is an effective divisor on S such that

1. For some $\epsilon > 0$, $(S, (\frac{m}{n} + \epsilon)C)$ is semi-log-canonical and the divisor $K_S + (\frac{m}{n} + \epsilon)C$ is ample.
2. $nK_S + mC \sim 0$.
3. $\chi(\mathcal{O}_S) = 1$.

References

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[3] Anand Deopurkar and Changho Han. "Stable log surfaces, admissible covers, and canonical curves of genus 4". In: *arXiv preprint arXiv:1807.08413* (2018).

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Main result

Definition. Let $\mathfrak{X}$ be the moduli stack parametrizing stable pairs of type $(1, 2)$ admitting a $\mathbb{Q}$-Gorenstein smoothing to a pair of the form (Σ_5, C). Let $\mathfrak{U} \subset \mathfrak{X}$ be the open substack parametrizing the following pairs (S, C):

1. S_{sing} consists of at worst a single $\frac{1}{20}(1, 9)$ cyclic quotient singularity; the curve C is smooth and avoids S_{sing}.
2. S_{sing} is of type $\frac{1}{4}(1, 1) \oplus A_1 \oplus A_2$ or $\frac{1}{4}(1, 1) \oplus A_4$; the curve C is as in 1.
3. S is the cone in $\mathbb{P}^5$ over an elliptic curve embedded in $\mathbb{P}^4$ via a degree 5 line bundle; the curve C is as in 1.
4. S is isomorphic to Σ_5, and C has at worst an A_{13} singularity.

Let $\mathfrak{U}^0 \subset \mathfrak{U} \subset \mathfrak{X}$ parametrize the first three types.

Theorem(G [4]).

1. The stack $\mathfrak{U}$ is smooth, Deligne-Mumford, and fits into the diagram

$$\begin{array}{ccccc} \tilde{\mathfrak{U}} & \xrightarrow{\nu} & \mathfrak{U} & & \\ {\scriptstyle \tilde{j}}\downarrow & \overset{j}{\swarrow} & & \overset{\tilde{\varphi}}{\searrow} & \\ \overline{\mathcal{M}}_6 & & \overset{\varphi}{\dashrightarrow} & & (D/\Gamma)^* \end{array}$$

   where j is the natural forgetting map and $\tilde{\varphi}$ extends the double cover construction of φ. Moreover, j restricts to a surjective morphism

$$j|_{\mathfrak{U}_0} : \mathfrak{U}^0 \twoheadrightarrow \mathcal{M}_6 \setminus \mathcal{H}$$

   where $\mathcal{H}$ denotes the hyperelliptic locus. The map j is also birational and restricts to an isomorphism over the locus of pairs of the form (Σ_5, C), where C is smooth and of class $-2K_{\Sigma_5}$. The stack $\tilde{\mathfrak{U}}$ and the morphisms $\tilde{j}$ and ν are well-understood: They are constructed via the simultaneous stable reduction process of Casalaina-Martin and Laza ([5]). The image of $\tilde{j}$ is a partial compactification of $\mathcal{M}_6$. The * on the target of φ denotes the Satake-Bailey-Borel compactification of the period domain.

2. The singularities that occur on surfaces in $\mathfrak{U}$ are precisely those in the following list:
   (a) All of the $\mathbb{Q}$-Gorenstein deformations of the cyclic quotient singularity $\frac{1}{20}(1, 9)$.
   (b) Singularities of type $\frac{1}{4}(1, 1) \oplus A_1 \oplus A_2$ and $\frac{1}{4}(1, 1) \oplus A_4$.
   (c) A simple elliptic singularity of degree 5.

Idea of proof

The proof involves explicit construction of stable pairs containing special curves.

Example. Given a plane quintic curve C, choose a line $\ell \subset \mathbb{P}^2$ transverse to C. Blow up the points of intersection of C and ℓ, and contract the strict transform of ℓ. This process produces a stable pair with a $\frac{1}{4}(1, 1)$ singularity containing C.

We verify by definition that these pairs are stable and moreover that their associated double branched covers are (degenerations of) $K3$ surfaces. For each pair, we must exhibit a $\mathbb{Q}$-Gorenstein smoothing to a pair (Σ_5, C) with C smooth and of class $-2K_{\Sigma_5}$ to verify membership in $\mathfrak{U}$. The existence of such smoothings is guaranteed by the following smoothability criterion.

Theorem(G [4]). Let Δ be a disk. Let (S, C) be a semi-log-canonical stable pair of type (m, n) such that S has T-singularities and C is Cartier. Then

1. The pair (S, C) is smoothable.
2. The generic fiber of any $\mathbb{Q}$-Gorenstein deformation of (S, C) over Δ is smoothable.
3. Any $\mathbb{Q}$-Gorenstein deformation of the singularities of S over Δ can be realized on a stable pair of type (m, n).