Degree of Irrationality of Very General Abelian Surfaces (2019)

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Motivations

Q: Given a projective variety X of dimension n, can we measure how far away it is from being rational?

- $n = 1$: The *gonality* of a curve C is defined to be the smallest degree of a branched covering $C' \to \mathbb{P}^1$ (where C' is the normalization of C).
- In higher dimensions, one generalization is the *degree of irrationality*, defined as:

$$\operatorname{irr}(X) = \min\left\{\delta > 0 \;\middle|\; \begin{array}{c}\exists \text{ degree } \delta \text{ rational dominant} \\ \text{map } X \dashrightarrow \mathbb{P}^n\end{array}\right\}$$

Hypersurfaces:

Theorem (BDELU [2], 2017)

Let $X \subset \mathbb{P}^{n+1}$ be a very general smooth hypersurface of dimension n and degree $d \geq 2n+1$. Then

$$\operatorname{irr}(X) = d - 1.$$

Furthermore, if $d \geq 2n+2$ then any rational map

$$f : X \dashrightarrow \mathbb{P}^n \quad \text{with} \quad \deg(f) = d - 1$$

is birationally equivalent to projection from a point of X.

Key Insight of [2]:

- The positivity of the canonical bundle can bound the gonality of curves contained in the hypersurface.

Q: Can we say anything about the degree of irrationality for varieties with trivial K_X?

Abelian surfaces

Let A be an abelian surface.

- From [1], we know that $\operatorname{irr}(A) \geq 3$.
- Yoshihara [9] proved that if

$$A \supset (\text{smooth curve } C \text{ of genus } 3),$$

then $\operatorname{irr}(A) = 3$.

Now consider a polarized abelian surface $(A, L) = (A_d, L_d)$ of type $(1, d)$ and assume that $\operatorname{NS}(A) \cong \mathbb{Z}[L]$. An argument of Stapleton [7] showed that there is a positive constant C such that

$$\operatorname{irr}(A) \leq C \cdot \sqrt{d}$$

for $d \gg 0$. It was conjectured in [2] that equality holds asymptotically, i.e.

$$\limsup_{d \to \infty} \operatorname{irr}(A_d) = \infty.$$

Our main result shows that this is maximally false:

Main Theorem (Chen, 2019)

For an abelian surface $A = A_d$ with Picard number $\rho = 1$, one has

$$\operatorname{irr}(A) \leq 4.$$

Set-up

Assuming as before that $\operatorname{NS}(A) \cong \mathbb{Z}[L]$:

- Numerically $L^2 = 2d$ and $h^0(L) = d$.
- Let $Z = \{\text{two-torsion points of } A\}$.
- Consider the space of *even* sections $H^0(A, \mathcal{O}_A(2L))^+$.

Key Ingredient

Even sections of $\mathcal{O}_A(2L)$ vanish to even order at <u>all</u> two-torsion points, so we need to impose at most

$$1 + 3 + \cdots + (2m - 1) = m^2$$

conditions for all even sections to vanish to order $2m$ at a given $p \in Z$.

- Fixing suitable multiplicities at all points $p \in Z$ and utilizing Lagrange's four-square theorem, we construct a subspace

$$V \subset H^0(A, \mathcal{O}_A(2L))^+ \iff \mathfrak{d} = \mathbb{P}_{\text{sub}}(V) \subseteq |2L|^+.$$

- By our choice of multiplicities, $\mathfrak{d}$ defines a rational map

$$\varphi : A \dashrightarrow S \subset \mathbb{P}^N$$

where S is a surface.
- We obtain bounds on $\deg S$ and $\deg \varphi$, and use these to construct a 4-to-1 rational dominant map $A \dashrightarrow \mathbb{P}^2$.

Main Difficulty:

- The linear system $\mathfrak{d}$ can have fixed components; this approach was inspired in part by the work of Bauer in [3], [4].

A modification shows that the assumption $\rho = 1$ can be weakened to include all simple abelian surfaces.

Related ideas

- Voisin [8] showed that the covering gonality of a very general abelian variety A of dimension n is bounded from below by $\approx \log(n)$. This lower bound was improved to $\lceil \frac{1}{2}n + 1 \rceil$ by Martin [6].
- What about computing the degree of irrationality for K3 surfaces? In the same paper [2], it is conjectured that for polarized K3 surfaces (S_d, B_d) of genus d, there exist positive constants C_1, C_2 such that

$$C_1 \cdot \sqrt{d} \leq \operatorname{irr}(S_d) \leq C_2 \cdot \sqrt{d}$$

for $d \gg 0$. As far as we can see, this remains plausable.
- There are many other measures of irrationality that would be interesting to explore, such as covering gonality and connecting gonality.

References

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