Transverse Lines to Surfaces over Finite Fields

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Abstract

In this work, we prove that when X is a smooth reflexive surface of degree d in a projective space $\mathbb{P}^3_{\mathbb{F}_q}$ satisfying $q \geq \frac{3+\sqrt{17}}{4}d$, there exists an $\mathbb{F}_q$-line transverse to X. This work is a 2-dimensional generalization of a previous result of S. Asgarli [Asg19].

Motivation

Given a smooth hypersurface $X \subseteq \mathbb{P}^n_k$ defined over an infinite field k, we have

Theorem.[Bertini's Theorem] *For a general choice of a hyperplane $H \subseteq \mathbb{P}^n_k$, $X \cap H$ is smooth.*

Corollary. *If X is a smooth hypersurface of degree d, then for a general choice of line $L \subset \mathbb{P}^n_k$, $X \cap L$ consists of d distinct points.*

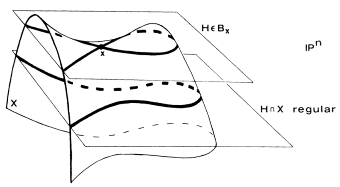

Figure: For a general hyperplane H, $X \cap H$ is smooth
(Picture is from: http://www.staff.uni-mainz.de/dfesti/AlgebraicGeometryIII.html).

Definition. Given a degree d smooth hypersurface $X \subseteq \mathbb{P}^n_k$, a k-line is called a k-transverse line to X if intersects X at d distinct (geometric) points.

Question 1

If k is a finite field $\mathbb{F}_q$, can we still find a $\mathbb{F}_q$-transverse line to X?

Definition. A hypersurface X is called *reflexive* if the Gauss map

$$\varphi : X \dashrightarrow X^* \subset (\mathbb{P}^n)^*$$

is separable, where X^* is the dual variety of X.

An Example

The answer to the Question 1 is **negative**.

Example 1. Let C be a smooth curve with $\deg(C) = q+2$ such that

$$\#C(\mathbb{F}_q) = \#\mathbb{P}^2(\mathbb{F}_q).$$

- Such curves exist by Homma and Kim [HK13].
- For such a curve C, every $\mathbb{F}_q$-line L intersects C at $q+2$ points in the algebraic closure of $\mathbb{F}_q$ (counted with multiplicity).
- $q+1$ of these points are already in $\mathbb{F}_q$ since $\#L(\mathbb{F}_q) = \#\mathbb{P}^1(\mathbb{F}_q)$.
- Hence all the $q+2$ intersection points are in $\mathbb{F}_q$. And the double intersection point is the tangent point of L at C.
- Thus all the $\mathbb{F}_q$-lines are tangent lines of C.

Our Goal

To remedy the original Bertini theorem in the case of finite fields, there are at least two approaches.

- We can consider the intersection of X with smooth curves of genus > 0. According to Poonen [Poo04], there exits plenty of smooth curves defined over $\mathbb{F}_q$ which intersect X transversely.
- Or we can ask the following question:

Question 2

Given a projective variety $X \subseteq \mathbb{P}^n$ defined over a finite field $k = \mathbb{F}_q$. Can we find a positive integer n such that for a field extension k'/k of degree $[k' : k] \geq n$, there exists a line over k' transverse to X? How would the minimal value of n depend on the invariants of X (e.g. the degree of X)?

Main Result

Suppose that X is a smooth reflexive surface of degree d in $\mathbb{P}^3$ over finite field $\mathbb{F}_q$ satisfying

$$q \geq \frac{3+\sqrt{17}}{4}d \approx 1.7808d$$

Then there is an $\mathbb{F}_q$-line in $\mathbb{P}^3$ which is transverse to X.

Remarks

- For reflexive smooth surfaces of degree d, we answer Question 2 by finding
$$n = \log_q(1.8 \cdot d).$$
A similar result [Asg19] for reflexive curves is:
$$n = \log_q(d-1).$$
- How sharp is our bound? According to computer experiments, there are smooth surfaces X of degree $q+2$ in $\mathbb{P}^3_{\mathbb{F}_q}$ such that $\#X(\mathbb{F}_q) = \#\mathbb{P}^3(\mathbb{F}_q)$. For these surfaces, every $\mathbb{F}_q$-line is a tangent line. So $n \geq \log_q(d-1)$ is necessary.

Mathematical Setups

- $X \subseteq \mathbb{P}^3$ is a smooth surface defined by a degree d homogeneous polynomial
$$F = F(X_0, X_1, X_2, X_3) \in \mathbb{F}_q[X_0, X_1, X_2, X_3].$$
- Take $F_i := \frac{\partial F}{\partial X_i}$ and
$$F_i^{(q)}(X_0, X_1, X_3, X_4) := F_i(X_0^q, X_1^q, X_2^q, X_3^q).$$
- Define the two surfaces
$$X_{1,0} := \{F_{1,0} = 0\} \quad \text{and} \quad X_{0,1} := \{F_{0,1} = 0\}$$
where
$$F_{1,0} := X_0^q F_0 + X_1^q F_1 + X_2^q F_2 + X_3^q F_3$$
$$F_{0,1} := X_0 F_0^{(q)} + X_1 F_1^{(q)} + X_2 F_2^{(q)} + X_3 F_3^{(q)},$$

Main Methods

The main idea in the proof of our theorems is counting the number of $\mathbb{F}_q$-tangent lines of a smooth degree d surface X.

1. Suppose all the $\mathbb{F}_q$-lines L_i are tangent to X; then they are classified into two types:
   a. At least one of the tangency points of L_i is defined over $\mathbb{F}_q$. We call tangent lines **rational tangents**.
   b. All the tangency points of L_i are not defined over the ground field $\mathbb{F}_q$. These tangent lines are called **special tangents**.
2. Under our assumption, we obtain
$$\#\{\text{rational tangents}\} + \#\{\text{special tangents}\} = \#\{\mathbb{F}_q\text{-lines in } \mathbb{P}^3\}$$
3. Every $\mathbb{F}_q$-point of X contributes $q+1$ rational tangents. So
$$\#\{\text{rational tangents}\} \leq \#X(\mathbb{F}_q) \cdot (q+1)$$
4. The tangency points of special tangent lines are all contained in the intersection
$$X \cap X_{0,1} \cap X_{1,0}.$$
5. On the other hand, the number of $\mathbb{F}_q$-lines in $\mathbb{P}^3_{\mathbb{F}_q}$ is $(q^2+1)(q^2+q+1)$.
6. Thus if $X \cap X_{0,1} \cap X_{1,0}$ is a finite set, we should have
$$\#X(\mathbb{F}_q) \cdot (q+1) + \#(X \cap X_{0,1} \cap X_{1,0}) \geq (q^2+1)(q^2+q+1)$$
which will lead to a contradiction if q is large enough. After some algebra, one can see that there exists a transverse line when $q \geq 1.537d$.
7. When $X \cap X_{0,1} \cap X_{1,0}$ contains curves, a similar but more complicated analysis produces the bound $q \geq \frac{3+\sqrt{17}}{4}d \approx 1.7808d$.

Main References

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