The Kawamata-Morrison-Totaro Cone Conjecture for Log Calabi-Yau Surfaces

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Background

For a smooth projective variety Y, the cone of curves of Y is defined by

$$\overline{\mathrm{Curv}}(Y) = \overline{\langle \sum a_i[C_i] \mid a_i \in \mathbb{R}_{>0} \text{ and curve } C_i \subset Y \rangle},$$

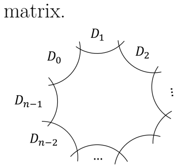
which is a subset of $H_2(Y, \mathbb{R})$. The dual of $\overline{\mathrm{Curv}}(Y)$ is the Nef cone, $\mathrm{Nef}(Y)$. In some cases (e.g., when Y is Fano), $\overline{\mathrm{Curv}}(Y)$ (and thus $\mathrm{Nef}(Y)$) is rational polyhedral, meaning it has finitely many generators. In some cases, the cone may be round:

rational polyhedral cone round cone

Morrison's cone conjecture states that for a smooth Calabi-Yau manifold Y, the automorphism group of Y acts on its effective Nef cone with rational polyhedral fundamental domain. Totaro generalized Morrison's conjecture to Kawamata log terminal (klt) Calabi-Yau pairs (Y, D). We are researching a version of the cone conjecture that is related to but different from Totaro's version.

Project Goal

We are studying pairs (Y, D) where Y is a smooth projective surface and $D = D_0 + D_1 + \cdots + D_{n-1}$ is a reduced normal crossing divisor on Y, with the properties that $K_Y + D = 0$ and D has negative-definite self-intersection matrix.



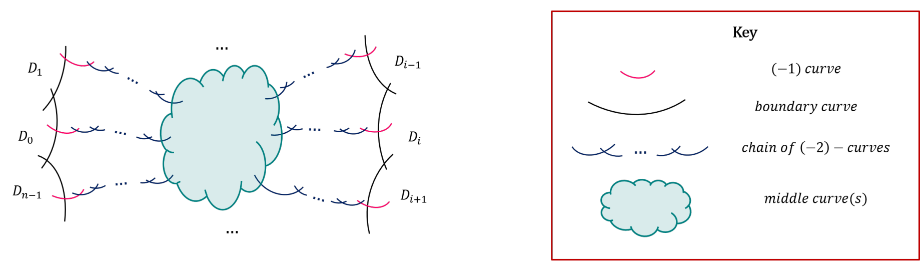
We consider the pair (Y, D) with a distinguished complex structure in which the mixed Hodge structure on $U = Y \backslash D$ is split. The goal of our project is to prove Morrison's cone conjecture in this special case.

Our project differs from Totaro's work because our pair (Y, D) is not klt, and we must consider (Y, D) with the special complex structure (otherwise, our conjecture is false, as observed by Totaro). For this complex structure, the cone $\overline{\mathrm{Curv}}(Y)$ is as large as possible.

Results and General Method

We have shown that the cone conjecture holds in our special case when D has at most six components ($n \leq 6$). In these cases, the Nef cone is rational polyhedral and we give explicit generators for its dual $\overline{\mathrm{Curv}}(Y)$. Any pair (Y, D) for fixed $n \leq 6$ is obtained as a blowup of a fixed pair $(\bar{Y}, \bar{D})$ - this is due to Looijenga for $n \leq 5$ and Simonetti for $n = 6$. We blow up smooth points of $\bar{D}$ and let D be its strict transform.

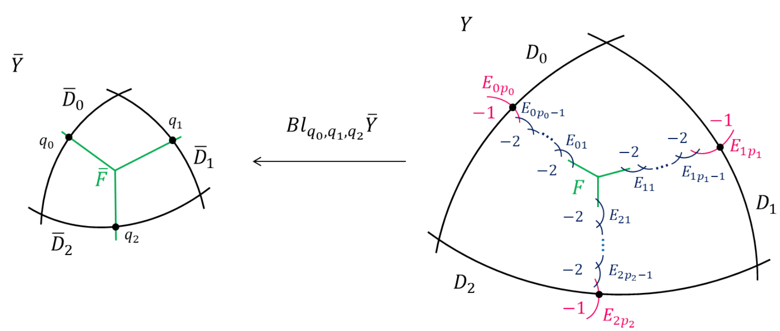
Our general method for each $n \leq 5$ is as follows. After arbitrarily many blowups at a chosen point q_i on each boundary curve $\bar{D}_i$, we obtained a collection of curves as shown below:



There is a basis for $\mathrm{Pic}(Y)$ which is the set of all (-1)-curves and all (-2)-curves (which consists of the exceptional curves and the middle curves). We then found expressions of the dual basis elements as positive linear combinations of the boundary curves and exceptional curves. The curves that appeared in these expressions, along with the middle curve(s), form the set of generators of $\overline{\mathrm{Curv}}(Y)$. For the case $n = 6$, the interior (-1)-curves and (-2)-curves do not form a basis of $\mathrm{Pic}(Y)$, but a similar method can be used to show that together with the boundary classes, they generate $\overline{\mathrm{Curv}}(Y)$.

Example: $n = 3$

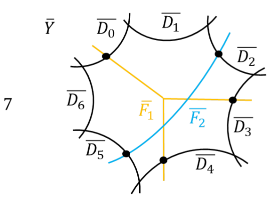
Here $\bar{Y} = \mathbb{P}^2$ and $\bar{D}$ is the toric boundary. The strict transform of $\bar{D}$ is the boundary $D = D_0 + D_1 + D_2$. The single middle curve F, which is the strict transform of a line $\bar{F}$ in $\mathbb{P}^2$, intersects three chains of (-2)-curves at one point. We blow up a total of p_i times at the point q_i on each boundary component $\bar{D}_i$.



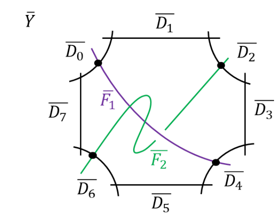
Work in Progress: $n = 7, 8, 9$

The case $n = 8$ splits into two subcases: $n = 8$(i) and $n = 8$(ii). When $n = 7$ and $n = 8$(ii) and in most cases of $n = 8$(i), we have shown that the automorphism group is infinite. In these cases, we get infinitely many automorphisms from the Mordell-Weil group of an elliptic fibration on Y, so $\mathrm{Nef}(Y)$ is not rational polyhedral. We need to prove that the action of the automorphism group on the Nef cone has a rational polyhedral fundamental domain.

$n = 7$



$n = 8$



Motivation

Results by Gross-Hacking-Keel on mirror symmetry for cusp singularities suggest we consider the pair (Y, D) with a distinguished complex structure. Under our conditions, there exists a contraction of (Y, D) to a cusp singularity (Y', p). Cusp singularities come in mirror dual pairs, and the embedding dimension m of the dual cusp is equal to $\max(n, 3)$, where n is the number of components of the boundary divisor D. By studying the Nef cone of (Y, D), we hope to give a description of the deformation space of the dual cusp, which is not well understood for m greater than six.

Acknowledgements

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