Effective divisors in the projectivized Hodge bundle

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Motivation

Computing effective divisor classes can reveal important information about the geometry of the underlying space. For example, in the 1980s Harris and Mumford computed the class of a certain Brill-Noether divisor $\mathcal{M}^1_{g,k} = \{[C] \in \mathcal{M}_g \mid \exists\, C \xrightarrow{k:1} \mathbb{P}^1\}$ (here $k = \frac{g+1}{2}$) to determine the Kodaira dimension of $\mathcal{M}_g$ for odd $g \geq 25$ [1]. The class W of the closure of the locus in $\overline{\mathcal{M}}_{g,1}$ of curves with a marked Weierstrass point was first calculated by Cukierman in [2]. The class of the divisorial stratum $\mathbb{P}\overline{\mathcal{H}}_g(2, 1^{2g-4})$ in $\mathrm{Pic}(\mathbb{P}\overline{\mathcal{H}}_g) \otimes \mathbb{Q}$ was computed in [3]. In general, computing the classes of certain geometrically defined effective divisors is quite helpful in determining the structure of the effective cone. Thus, we are led to ask:

Question: What effective divisor classes in $\mathbb{P}\overline{\mathcal{H}}_g$ are possible to compute? Are these divisors extremal in the pseudoeffective cone? (Ask me for further questions!)

Background and notation

- $\mathcal{H}_g$ is the Hodge bundle over $\mathcal{M}_g$ parametrizing pairs (C, ω) where C is a smooth genus g curve and ω is a holomorphic abelian differential on C. $\mathcal{H}_g(\mu)$ is the stratum consisting of (C, ω) where $\mu = (m_1, \ldots, m_n)$ is a partition of $2g-2$ describing the multiplicities of the zeros of ω.
- The Hodge bundle extends over the boundary of $\overline{\mathcal{M}}_g$, where the fiber over a nodal curve consists of stable differentials - that is, differentials that have at worst simple poles at the nodes with opposite residues on the two branches of the node.
- $\mathbb{P}\overline{\mathcal{H}}_g$ is the projectivization of this bundle and $\mathbb{P}\overline{\mathcal{H}}_g(\mu)$ the closure of the strata in $\mathbb{P}\overline{\mathcal{H}}_g$.
- $\overline{\mathcal{P}}(\mu)$ is the incidence variety compactification of the strata described in [4]. It is defined as follows: Let

$$\mathcal{P}(\mu) := \left\{(X, \omega, z_1, \ldots, z_n) \in \mathbb{P}\mathcal{H}_{g,n} \mid \operatorname{div}\, \omega = \sum_{i=1}^{n} m_i z_i\right\}$$

  where $\mathbb{P}\mathcal{H}_{g,n}$ denotes the projectivized Hodge bundle over $\mathcal{M}_{g,n}$ parametrizing pointed stable differentials (with ordered marked points). The incidence variety compactification $\overline{\mathcal{P}}(\mu)$ is defined to be the closure of $\mathcal{P}(\mu)$ inside $\mathbb{P}\overline{\mathcal{H}}_{g,n}$. In [4] a characterization of the boundary is given.
- $\mathrm{Pic}(\mathbb{P}\overline{\mathcal{H}}_g) \otimes \mathbb{Q} = \langle \eta, \lambda, \delta_0, \ldots, \delta_{\lfloor g/2 \rfloor} \rangle$ where $\eta := \mathcal{O}_{\mathbb{P}\overline{\mathcal{H}}_g}(-1)$ and the remaining classes are the pullbacks from $\overline{\mathcal{M}}_g$.
- $\overline{\mathcal{Q}}_g$ over $\overline{\mathcal{M}}_g$ is the bundle of quadratic differentials and $\overline{\mathcal{Q}}_g(\mu)$ is the stratum parametrizing quadratic differentials where μ, now a partition of $4g-4$, describes the multiplicities of the zeros.
- Let X be a projective variety. D is an extremal divisor in the pseudoeffective cone $\overline{\mathrm{Eff}}^1(X)$ if for any linear combination $D = D_1 + D_2$ with D_i pseudoeffective, D and D_i are proportional.

Statement of results

Theorem 1. Let D be the following divisor:

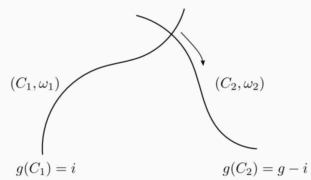
$$D := \overline{\left\{(C, \omega) \in \mathbb{P}\mathcal{H}_g \mid \operatorname{div}\, \omega \text{ contains a Weierstrass point}\right\}} \subset \mathbb{P}\overline{\mathcal{H}}_g.$$

In $\mathrm{Pic}(\mathbb{P}\overline{\mathcal{H}}_g) \otimes \mathbb{Q}$, the class $[D] = -(g-1)g(g+1)\eta + 2(3g^2+2g+1)\lambda - \frac{g(g+1)}{2}\delta_0 + \sum_{i=1}^{\lfloor g/2 \rfloor}(g+3)i(i-g)\delta_i$.

Theorem 2. The divisors $\mathbb{P}\overline{\mathcal{H}}_g(2, 1^{2g-4}) \subset \mathbb{P}\overline{\mathcal{H}}_g$ and $\mathbb{P}\overline{\mathcal{Q}}_g(2, 1^{4g-6}) \subset \mathbb{P}\overline{\mathcal{Q}}_g$ span extremal rays of the respective pseudoeffective cones.

Proof ideas and an example

The proof of Theorem 1 relies on the technique of test curves. The idea is to compute the intersection product of D with various curve classes in order to extract relations between the coefficients in the class formula.
Here is an example of a test curve in $\mathbb{P}\overline{\mathcal{H}}_g$ used in the calculation: fix general pointed curves $(C_1, q_1) \in \mathcal{M}_{i,1}$ and $(C_2, q_2) \in \mathcal{M}_{g-i,1}$, $1 \leq i \leq \lfloor g/2 \rfloor$, attached at a node $q_1 \sim q_2$, along with ω_1 and ω_2 general nonzero holomorphic differentials on C_1 and C_2 respectively, with zeros z_j, $1 \leq j \leq 2i-2$ and p_k, $1 \leq k \leq 2g-2i-2$. Now vary the point of attachment q_2 in C_2.



Computing the intersection product of the test curves with η, λ, and the boundary classes can be done by using known relations and the projection formula. To compute the intersection with D, we first pullback the test curves to the incidence variety compactification of the principal stratum, $\overline{\mathcal{P}}(1^{2g-2})$ and then pushforward to $\overline{\mathcal{M}}_{g,2g-2}$ via the morphism forgetting the differential. We define generalized Weierstrass divisors

$$D_{2g-2} := \overline{\left\{(X, \omega, z_1, \ldots, z_{2g-2}) \in \mathcal{P}(1^{2g-2}) \mid \text{some } z_i \text{ is a Weierstrass point}\right\}}$$

$$W_n := \overline{\left\{(C, z_1, \ldots, z_n) \in \mathcal{M}_{g,n} \mid \text{some } z_i \text{ is a Weierstrass point}\right\}}$$

in $\overline{\mathcal{P}}(1^{2g-2})$ and $\overline{\mathcal{M}}_{g,n}$, respectively, and we reduce the intersection calculation to one with W_{2g-2}. One complication is the presence of some collection of divisors E in the formula $\varphi^* W_{2g-2} = D_{2g-2} + E$ where $\varphi : \overline{\mathcal{P}}(1^{2g-2}) \to \overline{\mathcal{M}}_{g,2g-2}$. (Ask me to explain!)

Example. We will verify the above divisor class for a curve B consisting of a general pencil of plane quartics with canonical divisors given by a fixed general line $L \subset \mathbb{P}^2$. In genus 3, $D = -24\eta + 68\lambda - 6\delta_0 - 12\delta_1$. By standard calculations, we have that $B \cdot \lambda = 3$, $B \cdot \delta_0 = 27$, and $B \cdot \delta_1 = 0$, and $B \cdot \eta = 1$. So, $B \cdot D = 18$. On the other hand, we have that the degree of the curve in $\mathbb{P}^2$ traced out by the flex points of a general pencil of degree d curves is $6d-6$ [5]. Thus, we have verified that indeed $B \cdot D = 18$.

The proof of Theorem 2 relies on the following lemma from [6] :

Lemma. ([6] Proposition 4.1) Suppose D is an irreducible effective divisor and A is a big divisor in a projective variety X. Let S be a set of irreducible effective curves contained in D such that the union of these curves is Zariski dense in D. If for every curve C in S we have

$$C \cdot (D + dA) \leq 0$$

for a fixed $d > 0$, then D is an extremal divisor in the pseudoeffective cone $\overline{\mathrm{Eff}}^1(X)$.

The collection of Teichmüller curves generated by some $(X, \omega) \in \mathbb{P}\overline{\mathcal{H}}_g(2)$ and $(X, q) \in \mathbb{P}\overline{\mathcal{Q}}_g(2)$ is Zariski dense in the respective strata and satisfy the intersection condition in the lemma. To show the latter point, we use the known classes of $\mathbb{P}\overline{\mathcal{H}}_g(2)$ and $\mathbb{P}\overline{\mathcal{Q}}_g(2)$ computed in [3] and [7] respectively, as well as the intersection data provided in [8] and [9].

References

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