On stringy Euler characteristics of Clifford non-commutative varieties

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Abstract

It was shown by Kuznetsov that complete intersections of n generic quadrics in $\mathbb{P}^{2n-1}$ are related by Homological Projective Duality to certain non-commutative (Clifford) varieties which are in some sense birational to double covers of $\mathbb{P}^{n-1}$ ramified over symmetric determinantal hypersurfaces.
Mirror symmetry predicts that the Hodge numbers of the complete intersections of quadrics must coincide with the appropriately defined Hodge numbers of these double covers. We observe that these numbers must be different from the well-known Batyrev's stringy Hodge numbers, else the equality fails already at the level of Euler characteristics. We define a natural modification of stringy Hodge numbers for the particular class of Clifford varieties, and prove the corresponding equality of Euler characteristics in arbitrary dimension.

Complete quadrics

Let $\dim V = n$ and $\Phi = \mathbb{P}\mathrm{Sym}^2 V^*$. Consider the successive blowups along Φ_{n-1}, the loci of *quadrics* of corank $\geq n-1$ in Φ, then the proper preimage of Φ_{n-2}, the loci of quadrics of corank $\geq n-2$ in Φ, etc. The resulting space is the space of complete quadrics, which is a smooth variety $\hat{\Phi}$ which parametrizes the flags

$$0 = F^0 \subset F^1 \subset \ldots \subset F^{l-1} \subset F^l = V$$

together with non-degenerate quadrics $\mathbb{C}q_i \in \mathbb{P}\mathrm{Sym}^2(F^{i+1}/F^i)^*$. The map $\pi : \hat{\Phi} \to \Phi$ is given by interpreting a quadratic from on F^l/F^{l-1} as a quadratic form on $F^l = V$. See [1, 3]. The exceptional divisors D_i, $2 \leq i \leq n-1$ and D_1 (the proper preimage of Φ_1) form a simple normal crossing divisor.

Double cover

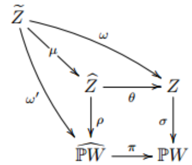
Let $\dim V = 2n$. We choose a dimension n subspace $W \subset \mathrm{Sym}^2 V^*$ in general position with bases $(q_1, \ldots, q_n)$ and consider $\mathbb{P}W$. Let $\phi_i = \Phi_i \cap \mathbb{P}W$. The double cover $\sigma : Z \to \mathbb{P}W$ ramified in ϕ_1 can be written in coordinates as a hypersurface in a weighted projective space

$$y^2 = \det\left(\frac{\partial^2}{\partial x_i \partial x_j} \sum_1^n u_k q_k(\overline{x})\right)_{i,j}$$

where y has weight n and u_i have weight one. The restriction $\pi : \widehat{\mathbb{P}W} = \pi^{-1}(\mathbb{P}W) \to \mathbb{P}W$ is a log-resolution of $\mathbb{P}W$. Let $T_i = D_i \cap \widehat{\mathbb{P}W}$, $i = 1, 2, \ldots, k$.

Discrepancies

Denote by $\widehat{Z}$ the normalization of the fiber product of $\pi : \widehat{\mathbb{P}W} \to \mathbb{P}W$ and $\sigma : Z \to \mathbb{P}W$. Let E_i be the proper preimages of T_i under ρ for $1 \leq i \leq k$.



The singularities of $\widehat{Z}$ are toroidal and are amenable to usual toric resolution techniques. We obtain the resolution $\mu : \widetilde{Z} \to \widehat{Z}$. The exceptional divisors of the resolution of singularities of Z, $\omega = \theta \circ \mu$, are $\widetilde{E}_\diamondsuit$, where $\diamondsuit \in S = \{i | 1 \leq i \leq k\} \cup \{(i,j) | 1 \leq i < j \leq k$ and i, j are odd$\}$. The discrepancies of $\widetilde{E}_\diamondsuit$ are $d_\diamondsuit = i^2 - 1$, if $\diamondsuit = i, 1 \leq i \leq k$, i is odd; $d_\diamondsuit = \frac{i^2}{2} - 1$, if $\diamondsuit = i, 1 \leq i \leq k$, i is even; $d_\diamondsuit = \frac{1}{2}(i^2 + j^2) - 1$, if $\diamondsuit = (i,j) \in S$. Define the modified discrepancies to be

1. $a_\diamondsuit = d_\diamondsuit$, if $\diamondsuit = i, 1 \leq i \leq k$,
2. $a_{i,j} = d_{i,j} + \frac{3}{2}(j - i)$, if $\diamondsuit = (i,j) \in S$.

Clifford-stringy Euler characteristics

1. Define Clifford-stringy Euler Characteristics of Z:
$$\chi_{cst}(Z) = \sum_{J \subseteq S} \chi(\widetilde{E}_J^\circ) \prod_{\diamondsuit \in J} \frac{1}{1 + a_\diamondsuit},$$
where χ denotes Euler characteristics with compact support and $\widetilde{E}_J^\circ = \cap_{\diamondsuit \in J} \widetilde{E}_\diamondsuit - \cup_{\diamondsuit \in S-J} \widetilde{E}_\diamondsuit$.
2. For $y \in Z$, define the local contributions of y to $\chi_{cst}(Z)$:
$$S_{cst}(y) = \sum_{J \subseteq S} \chi(\widetilde{E}_J^\circ \cap \omega^{-1}(y)) \prod_{\diamondsuit \in J} \frac{1}{1 + a_\diamondsuit}.$$
3. $\chi_{cst}(Z) = \Sigma_i \chi(Z_i) S_{cst}(y \in Z_i)$, where $Z = \sqcup_i Z_i$ is the stratification of Z into the sets on which S is constant.

Let $\phi_{t,Z}$ be the preimage of ϕ_t under the map σ and $\phi_{t,Z}^\circ = \phi_{t,Z} - \phi_{(t-1),Z}$, where $t \in \{1, 2, \ldots, k\}$.

Proposition

The local contribution of $y \in \phi_{t,Z}^\circ$ to the Clifford-stringy Euler characteristics of Z is

1. $S_{cst}(y \in \phi_{t,Z}^\circ) = 1$, if $1 \leq t \leq k, t$ is odd;
2. $S_{cst}(y \in \phi_{t,Z}^\circ) = 2$, if $1 \leq t \leq k, t$ is even.

Theorem

Let $Y = \{v \in V | q(v) = 0$ for all $q \in \mathbb{P}W\}$ be the complete intersection associated to W in $\mathbb{P}V \cong \mathbb{P}^{2n-1}$. Then we have

$$\chi_{cst}(Z) = \chi(Y).$$

Sketch of proof

Consider the universal quadric $\mathbb{H} = \{(q,v) \in \mathbb{P}W \times \mathbb{P}V | q(v) = 0\}$ and calculate its Euler characteristics by two projection.

1. By projection $\eta : \mathbb{H} \to \mathbb{P}V$ such that $\eta((q,v)) = v$ for $(q,v) \in \mathbb{H}$, we obtain
$\chi(\mathbb{H}) = \chi(\mathbb{P}^{n-1})\chi(Y) + \chi(\mathbb{P}^{n-2})\chi(\mathbb{P}V - Y) = 2n(n-1) + \chi(Y)$.
2. By projection $\eta' : \mathbb{H} \to \mathbb{P}W$ such that $\eta'((q,v)) = q$ for $(q,v) \in \mathbb{H}$, we obtain
$\chi(\mathbb{H}) = \Sigma_{0 \leq t \leq k} \chi(Q_t) \chi(\phi_t^\circ) = (2n-2)n + \chi_{cst}(Z)$.

Comments and open questions

Let $\widetilde{X}$ be the blowup of X with center Y, where Y is a locally complete intersection subscheme of codimenion c. Theorem 1.6 in [2] shows that the numbers of components in the semiorthogonal decomposition of $D^b(coh(\widetilde{X}))$ is related to the discrepancy $c-1$ of the blowup. Thus we get the inspiration that the meaning of discrepancy changes we made to define Clifford-stringy Euler characteristics are expected to be interpreted in terms of the Lefschetz decompositions of derived categories.
The main theorem of this paper can be conjecturally generalized to a statement on Hodge polynomials or elliptic genera.

References

[1] A. Bertram, *An application of a log version of the Kodaira vanishing theorem to embedded projective varieties*, preprint alg-geom/9707001.

[2] L. Borisov, Z. Li, *On Clifford double mirrors of toric complete intersections*, Adv. Math. 328 (2018) 300-355.

[3] M. Thaddeus, *Complete collineations revisited.* Math. Ann. 315 (1999), no. 3, 469-495.