On stringy Euler characteristics of Clifford non-commutative varieties

Lev Borisov, Chengxi Wang
Rutgers University - New Brunswick

Abstract

It was shown by Kuznetsov that complete intersections of n generic quadrics in $\mathbb{P}^{2n-1}$ are related by Homological Projective Duality to certain non-commutative (Clifford) varieties which are in some sense birational to double covers of $\mathbb{P}^{2n-1}$ ramified over symmetric determinantal hypersurfaces. Mirror symmetry predicts that the Hodge numbers of the complete intersections of quadrics must coincide with the appropriately defined Hodge numbers of these double covers. We observe that these numbers must be different from the well-known Batyrev’s stringy Hodge numbers, else the corresponding equality of Euler characteristics fails already at the level of Euler characteristics. The singularities of $\tilde{Z}$ are toroidal and are amenable to usual toric resolution techniques. We obtain the resolution $\mu : Z \rightarrow \tilde{Z}$. The exceptional divisors of the resolution of singularities of $Z$, $\omega = \theta S Y$, are $E_0$, where $\theta \in S = \{ i | 1 \leq i \leq k \}$. The discrepancies of $E_0$ are $d_0 = i^2 - 1$, $d_0 = i$, $1 \leq i \leq k$, $i$ is odd. $d_0 = \frac{i}{2}(i^2 + j^2) - 1$, if $\theta = (i, j) \in S$. Define the modified discrepancies to be:

- $q_0 = d_0$, if $\theta = i$, $1 \leq i \leq k$,
- $a_{ij} = d_{ij} + \frac{1}{2}(j - i)$, if $\theta = (i, j) \in S$.

Discrepancies

Denote by $\tilde{Z}$ the normalization of the fiber product of $\pi : \mathbb{P}W \rightarrow \mathbb{P}V$ and $\sigma : \mathbb{Z} \rightarrow \mathbb{P}W$. Let $E_i$ be the proper preimage of $T_i$ under $\mu$ for $1 \leq i \leq k$.

The singularities of $\tilde{Z}$ are toroidal and are amenable to usual toric resolution techniques. We obtain the resolution $\mu : Z \rightarrow \tilde{Z}$. The exceptional divisors of the resolution of singularities of $Z$, $\omega = \theta S Y$, are $E_0$, where $\theta \in S = \{ i | 1 \leq i \leq k \}$. The discrepancies of $E_0$ are $d_0 = i^2 - 1$, $d_0 = i$, $1 \leq i \leq k$, $i$ is odd. $d_0 = \frac{i}{2}(i^2 + j^2) - 1$, if $\theta = (i, j) \in S$. Define the modified discrepancies to be:

- $q_0 = d_0$, if $\theta = i$, $1 \leq i \leq k$,
- $a_{ij} = d_{ij} + \frac{1}{2}(j - i)$, if $\theta = (i, j) \in S$.

Clifford-stringy Euler characteristics

Define Clifford-stringy Euler Characteristics of $Z$:

$$\chi_{c,s}(Z) = \sum_{\theta} \chi(E_0) \prod_{i \subseteq J} \frac{1}{1 + a_{ij}},$$

where $\chi$ denotes Euler characteristics with compact support and $E_0 = \mu_{\theta \subseteq J} E_0 - \mu_{\theta \subseteq S \setminus J} E_0$.

For $y \in Z$, define the local contributions of $y$ to $\chi_{c,s}(Z)$:

$$S_{\mu}(y) = \sum_{\mu_{\theta \subseteq J}} \chi(E_0) \prod_{i \subseteq J} \frac{1}{1 + a_{ij}},$$

$$\chi_{c,s}(Z) = \sum_{\mu_\theta} \chi(Z_\theta) S_{\mu}(y \in Z_\theta),$$

where $Z_\theta = \mu_\theta Z$ is the stratification of $Z$ into the sets on which $S$ is constant.

Let $\phi_\theta Z$ be the preimage of $\phi_\theta$ under the map $\sigma$ and $\phi_{\theta^*} Z = \phi_\theta Z - \phi_{\theta^*} Z, t \in \{ 1, 2, \ldots, k \}$.

Proposition

The local contribution of $y \in \phi_{\theta^*} Z$ to the Clifford-stringy Euler characteristics of $Z$ is:

- $S_{\mu}(y \in \phi_{\theta^*} Z) = 1$, if $1 \leq t \leq k$, $t$ is odd;
- $S_{\mu}(y \in \phi_{\theta^*} Z) = 2$, if $1 \leq t \leq k$, $t$ is even.

Theorem

Let $Y = \{ v \in V | q(v) = 0 \}$ and $\chi_{c,s}(Z) = \chi(Y)$.

Sketch of proof

Consider the universal quadric $\mathbb{H} = \{(q, v) \in \mathbb{P}V \times \mathbb{P}V | q(v) = 0 \}$ and calculate its Euler characteristics by two projection.

- By projection $q : \mathbb{H} \rightarrow \mathbb{P}V$ such that $q(v) = v$ for $(q, v) \in \mathbb{H}$, we obtain $\chi(\mathbb{H}) = \chi(\mathbb{P}V - Y) = 2n(n-1)+\chi(Y)$.

- By projection $q(q, v) : \mathbb{H} \rightarrow \mathbb{P}V$ such that $q'(q, v) = q$ for $(q, v) \in \mathbb{H}$, we obtain $\chi(\mathbb{H}) = \chi(\mathbb{P}V) \chi(Q') = (2n-2)n + \chi_{c,s}(Z)$.

Comments and open questions

Let $\mathbb{X}$ be the blowup of $X$ with center $Y$, where $Y$ is a locally complete intersection subscheme of codimension $c$. Theorem 1.6 in [2] shows that the numbers of components in the seminvariant decomposition of $D(\text{coh}(\mathbb{X}))$ is related to the discrepancy $c - 1$ of the blowup. Thus we get the inspiration that the meaning of discrepancy changes we made to define Clifford-stringy Euler characteristics are expected to be interpreted in terms of the Lefschetz decompositions of derived categories. The main theorem of this paper can be conjecturally generalized to a statement on Hodge polynomials or elliptic genera.

References