Thursday, March 21, 2019 1:36 PM

joint w/ Junwu Tu, arXiv: 1706.09912, arXiv: 1810.05179, upcoming

in memoriam Boris Dubrovin (1950–2019)

1). What are we trying to accomplish?

<u>Problem</u>: Can we extract Gromov–Witten invariants of a space $\check{X}$ (counting curves in $\check{X}$ with prescribed topological invariants) directly from the Fukaya category of $\check{X}$?

<u>Why</u>? GW invariants hard to compute. But $\mathrm{Fuk}(\check{X})$ also hard so... but HMS predicts $\mathrm{Fuk}(\check{X}) \simeq D^b_{coh}(X)$ for some mirror space X, and $D^b_{coh}(X)$ easier to compute.

Kontsevich conjectured that HMS $\Rightarrow$ CMS...
If we had a definition of GW inv'ts that only use as input a category $\mathcal{C}$, we could define the invariants in other settings: B-model; orbifolds; LG-models, etc.

What is known? For $g=0$, Ganatra–Perutz–Sheridan (2015) following Barannikov–Kontsevich ($\sim$ 2000)

For $g \geq 0$, Costello (2005) + Kontsevich–Soibelman gave definition, not amenable to computation. (Requires making certain choices impossible to compute.)

New approach by (C–Tu) following an idea of Costello gives new definition which is amenable to computation.

<u>Thm</u>: (C–Tu 2017, C–Li–Tu 2018) For $g=1$, $n=1$, the new invariants give the expected result for elliptic curves.
Also for categories of matrix factorizations of A_n singularities.

<u>Note</u>: This is a computation in the B-model! (Uses the description, due to Polishchuk, of an A_∞-model for the derived category of the elliptic curve $E_\tau = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$)

The same computations are likely to be doable for $g=2$, $n=1$.

2). What are classical GW invts?

Start with a $\underset{\text{smooth cpt}}{}$ space X. Fix g, n, β. For $\alpha_1, \ldots, \alpha_n \in H^*(X)$, $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$
$\rightsquigarrow \langle \tau_{a_1}(\alpha_1), \ldots, \tau_{a_n}(\alpha_n)\rangle^{g,n}_{X,\beta} \in \mathbb{C}$

Think of this as a function

$$F_{g,n}(X) : \mathrm{Sym}^*(H^*(X)[[u]]) \longrightarrow \mathbb{C}$$

or dually

$$F(X) \in \mathrm{Sym}^*(u^{-1} H^*(X)[u^{-1}])((\hbar))[[\lambda]]$$
(genus $\hbar$, $\lambda \leftrightarrow X$)

Important note: The definition requires integration of cohomology classes on the <u>compactified</u> moduli spaces of curves. Equivalently, we construct cohomology classes on $\overline{M}_{g,n}$ and integrate against the fundamental class.

3). What is this in categorical language?

If $\mathcal{C} = \mathrm{Fuk}(\check{X})$, $HH_*(\mathcal{C}) = H^*(X)$. So we'd like to define, out of a category $\mathcal{C}$ (with extra structure) an invariant

$$F(\mathcal{C}) \in \mathrm{Sym}^*(u^{-1} HH_*(\mathcal{C})[u^{-1}]).$$

Extra structure:

- $\mathcal{C}$ will need to be smooth, proper, CY
- this guarantees that Hodge–de Rham s.s. degenerates

$$HC^-_*(\mathcal{C}) \simeq u^{-1} HH_*(\mathcal{C})[u^{-1}] \qquad (\text{same } u!)$$

but no specific isomorphism is chosen
- need to choose a splitting of H–dR (iso as above).

For $\mathrm{Fuk}(\check{X})$ there is a preferred choice of such splitting and using it will give the classical GW inv'ts. For $D^b_{coh}(X)$ need to pick one that matches this choice on the A-side.

4). Main ingredients for construction:

- do the construction at chain level, on $\mathrm{Sym}(u^{-1} C_*(A)[u^{-1}])$

Note that $\mathrm{Sym}(u^{-1} C_*(A)[u^{-1}])$ carries operators $b, B, \Delta, \{-,-\}$ (will explain later) which together form a BV structure.

- could try to find a "differential equation" that F will satisfy, hope this will determine F from initial conditions.

QME:

$$(b+uB)\tilde{S} + \hbar\Delta\tilde{S} + \frac{1}{2}\{\tilde{S},\tilde{S}\} = 0.$$
$$\Updownarrow$$
$$(b+uB+\hbar\Delta)\exp(\tilde{S}/\hbar) = 0$$

<u>Problems</u>: a). solution $\tilde{S}$ will generally not be uniquely determined by a few initial conditions
b). we'll see that such an $\tilde{S}$ corresponds to integrating only along a part of $\overline{M}_{g,n}$.

- where do all these operators come from?

<u>Thm</u> (Kontsevich–Soibelman, Costello): There is an action of the dg-PROP $C_*(M^{fr}_{g,n})^+$ on $C_*(A)$ for a cyclic A_∞-algebra A.

Explanations: - $M^{fr}_{g,n}$ - framed moduli spaces
- cyclic A_∞-algebra: encodes the category $\mathcal{C}$ up to Morita equivalence
- $b \leftrightarrow$ chain bdry
- $B \leftrightarrow$ rotate framing
- $\Delta \leftrightarrow$ sew  w/ twist
- $+$: ≥ 1 input (from ribbon graphs)

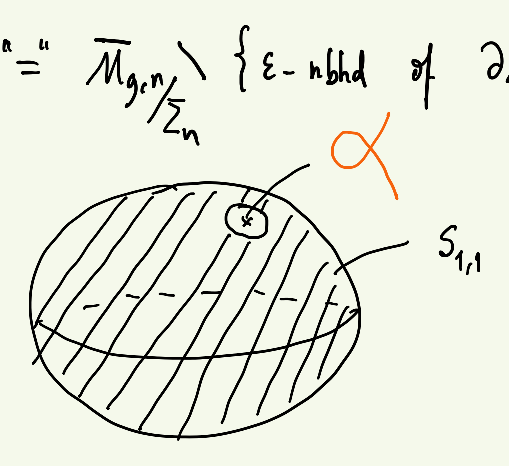
- In $C_*(M_{g,n}/\Sigma_n)$ the QME <u>does</u> have a unique solution S (up to homotopy) – discovered by Zwiebach–Sen, Costello; determined by

$$S_{0,3} = \frac{1}{6}[M_{0,3}/\Sigma_3].$$

What are the $S_{g,n}$'s? Think of them as

$$S_{g,n} \text{ "=" } \overline{M}_{g,n}/\Sigma_n \setminus \{\varepsilon\text{-nbhd of } \partial\overline{M}_{g,n}/\Sigma_n\}$$

<u>E.g.</u>



So we could take $\tilde{S}$ = image of S in

$$\mathrm{Sym}(u^{-1} C_*[u^{-1}])$$

Two new problems:

a) Not a problem to go from $C_*(M^{fr}_{g,n})$ to $C_*(M_{g,n})$ – take $(S^1)^n$ coinvariants
But the "+" <u>is</u> a problem: the operator Δ is defined by summing over all choices of two marked points, and these have to be outputs; after symmetrization, <u>all</u> markings would have to be outputs, and so Δ can not be defined in a way compatible with the action of the PROP.

b). Even if we succeeded in defining $\tilde{S}$, this would not behave well: it gives rise to a homology class $[\exp\tilde{S}]$ for $(b+uB+\hbar\Delta)$ not for $(b+uB)$ so we cannot pass to $(b+uB)$-homology to end up in

$$\mathrm{Sym}(HC^-(A)) \overset{\text{H-dR}}{\cong} \mathrm{Sym}(u^{-1} HH_*(A)[u^{-1}])$$

This is because we only "integrated" along the $S_{g,n}$'s not along the whole $\overline{M}_{g,n}$.

5) Construction;

1) Use an old idea of Costello:

Replace $C_*(M^{fr}_{g,n}/\Sigma_n)$ by the "Koszul resolution" $K_{g,n}$:

$$0 \to C_*(M^{fr}_{g,1,n-1}) \xrightarrow{\iota} C_*(M^{fr}_{g,2,n-2}) \xrightarrow{\iota} \cdots \xrightarrow{\iota} C_*(M^{fr}_{g,n,0}) \to 0$$

where $M^{fr}_{g,k,n-k}$ = moduli sp. of curves of genus g, k antisymmetric inputs, $n-k$ symmetric outputs

Then the operators b, uB, Δ, $\{-,-\}$ can be defined on the resolution s.t.

$$(C_*(M_{*,*}/\Sigma_n), d+\hbar\Delta, \{-,-\}) \simeq (K_{*,*}, b+uB+\iota+\hbar\Delta, \{-,-\})$$

as dgLa's. So we can solve the QME in $K_{g,n}$, for which <u>there is</u> a combinatorial model using ribbon graphs.

This solves the first problem, and we end up with elements

$$\tilde{S}_{g,n} \in \mathrm{Hom}(C_*(A)[[u]], u^{-1}C_*(A)[u])$$

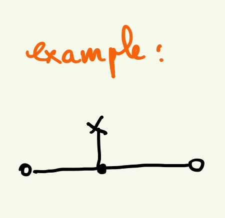
2) Use the splitting of Hodge–de Rham to trivialize the circle action, and in particular to trivialize the dgLa (explicitly)

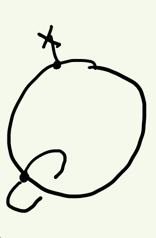
$$(K_{*,*}, b+uB+\iota+\hbar\Delta, \{-,-\}) \simeq (K_{*,*}, b+\iota)$$

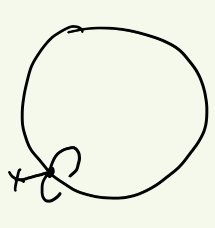
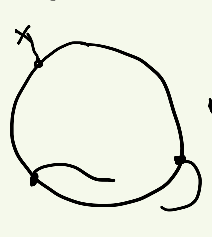
The homology of $(K_{*,*}, b+\iota)$ is $\mathrm{Sym}(u^{-1} HH_*(A)[u^{-1}])$

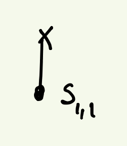
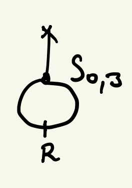
Map $\sum \tilde{S}_{g,n}$ under this trivialisation to get F.

6). Explicit example:

$S_{0,3} = \frac{1}{2}$ 

QME: $(b+uB)S_{1,1} = \Delta(S_{0,3}) = \frac{1}{2}$ 

$\Rightarrow S_{1,1} = \frac{1}{4}$  $+ \frac{1}{24}$  u^{-1}

$F_{1,1} =$  $+$ 

(evaluating using an A_∞ model for $D^b_{coh}(E_\tau)$ due to Polishchuk gives the correct answer, if we use the correct splitting required by Mirror Symmetry)