Pre-talk: NC geometry and invariants

Modern trend: replace varieties by their derived categories

$X \rightsquigarrow D^b_{coh}(X)$

and study properties of X that can be extracted from $D^b_{coh}(X)$.

Similar to old idea: replace space by ring of functions.
- problem: not all varieties are affine

Fact (true in more generality): If X is smooth projective, there exists a (non-unique, non-commutative) "algebra" A such that

$$D^b_{coh}(X) \simeq D^b(A\text{-mod})$$

So from this point of view, "all varieties are $\overset{\text{dg-}}{\text{affine}}$".

"algebra" = dg- or A_∞-algebra

What is an A_∞-algebra? Homotopy version of algebra.

graded Vector space A
+ operations $m_k : A^{\otimes k} \to A[2-k]$

Satisfying

$$(*_n) \quad \sum_{i+j+k=n} \pm\, m_{i+k+1}(x_1, \dots, x_i, m_j(x_{i+1}, \dots, x_{i+j}), x_{i+j+1}, \dots, x_{i+j+k}) = 0$$

for every n.

E.g. If $m_1 = 0$, $m_2(x, m_2(y,z)) = m_2(m_2(x,y), z)$
(assoc.)

E.g. If $m_k = 0$ $k \geq 3$, dg-algebra

How to construct A from X?

Pick a special vector bundle E on X (e.g. $\mathcal{O} \oplus \mathcal{O}(1) \oplus \dots \oplus \mathcal{O}(\dim X)$) called a generator and compute

$$A = \text{dg-Hom}_X(E, E)$$

using an (injective or Dolbeault) resolution of E.

Very big; can replace it by an equivalent A_∞ structure on the homology by a procedure known as Homological Perturbation.

Calabi-Yau / cyclic structure:

If X is CY there is a pairing on A:
$\langle -,- \rangle : A \otimes A \to k[-\dim X]$
such that the operations m_k are cyclic:
$\langle m_k(x_1, \dots, x_k), x_{k+1} \rangle = \pm \langle m_k(x_{k+1}, x_1, \dots, x_{k-1}), x_k \rangle$

Such A_∞ algebra + pairing is called cyclic.

What information about X can we recover from $D^b(X)$ or A?
- differential forms on X: $H^p(X, \Omega^q) \sim HH_*(X)$
- algebraic de Rham cohomology of X: $H^*_{dR}(X) \sim HP_*(X)$

Define $C_*(A) := A^{\otimes_k (*+1)}$

Differential $b: \dots \to A \otimes A \otimes A \to A \otimes A \to A \to 0$
$x|y|z \mapsto xy|z - x|yz + zx|y$
$a|b \mapsto ab - ba$

If have higher m_k, use them. (need internal degree)

Ex: the map $(C_*(A), b) \to (\Omega^*(A), 0)$
$a_0|a_1 \dots |a_n \mapsto a_0\, da_1 \wedge \dots \wedge da_n$
is a qiso if A is regular, commutative, over k of char 0.

We think of $HH_*(A)$ as NC-differential forms.

Cyclic homology:
b had homological degree -1
Define B of homological degree $+1$: (Connes)

$$B(a_0|a_1|\dots|a_n) = 1|a_0|\dots|a_n - 1|a_n|a_0|\dots|a_{n-1} + \dots$$
$$\pm 1|a_1|\dots|a_n|a_0$$

On $(C_*(A)((u)), b+uB) \rightsquigarrow HP_*(A)$
u-degree (-2) homological

Variants: $(C_*(A)[[u]], b+uB) \rightsquigarrow HC^-_*(A)$
$(C_*(A)((u))/C_*(A)[[u]], b+uB) \rightsquigarrow HC_*(A)$