Math 797W Homework 4

Paul Hacking

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We work over an algebraically closed field k.

(1) Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X, and $p \in X$ a point. Recall the definition of the *stalk* $\mathcal{F}_p$ of $\mathcal{F}$ at p:

$$\mathcal{F}_p = \varinjlim_{p \in U} \mathcal{F}(U) = \{(U, s) \mid U \subset X \text{ open}, p \in U, \text{ and } s \in \mathcal{F}(U)\}/\sim$$

where $(U, s) \sim (V, t)$ if $s|_W = t|_W$ for some open set $W \subset U \cap V$ such that $p \in W$.

(a) Let A be an abelian group. Define the associated *constant sheaf* $\underline{A}$ on a topological space X by

$$\Gamma(U, \underline{A}) = \{f \mid f \colon U \to A \text{ continuous }\}$$

for $U \subset X$ an open set, where we give A the discrete topology. Show that $\Gamma(U, \underline{A}) = A^I$ where I is the set of connected components of U. For $p \in X$ a point show that the stalk $\underline{A}_p = A$.

(b) Let X be a topological space, $Y \subset X$ a closed subset, and $\mathcal{F}$ a sheaf on Y. Let $i_*\mathcal{F}$ be the pushforward of $\mathcal{F}$ by the inclusion $i \colon Y \to X$, that is, $\Gamma(U, i_*\mathcal{F}) = \mathcal{F}(i^{-1}U) = \mathcal{F}(U \cap Y)$ for $U \subset X$ an open set. Show that $(i_*\mathcal{F})_p = \mathcal{F}_p$ if $p \in Y$ and $\mathcal{F}_p = \{0\}$ otherwise.

(c) Let X be a Riemann surface (a complex manifold of dimension 1) with its Euclidean topology, and $\mathcal{O}_X$ the sheaf of holomorphic functions on X. For $p \in X$ a point show that the stalk $\mathcal{O}_{X,p}$ is isomorphic to the ring $\mathbb{C}\{z\}$ of complex power series in a variable z with positive radius of convergence.

(d) Let X be an algebraic variety and $\mathcal{O}_X$ the sheaf of regular functions. Let $p \in X$ a point and $U \subset X$ an open affine subvariety such that $p \in U$. Show that the stalk $\mathcal{O}_{X,p}$ coincides with the local ring of X at p given by the localization $k[U]_{I(p)}$ of the coordinate ring $k[U] = \Gamma(U, \mathcal{O}_X)$ of U at the maximal ideal $I(p)$ corresponding to p.

(2) Recall that we say a sequence of sheaves

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

of abelian groups on a topological space X is *exact* if the sequence of stalks $\mathcal{F}'_p \to \mathcal{F}_p \to \mathcal{F}''_p$ is exact for all $p \in X$.

(a) Let X be a complex manifold. Let $\underline{\mathbb{Z}}$ be the constant sheaf on X with stalk $\mathbb{Z}$ (see Q1(a)), $\mathcal{O}_X$ the sheaf of holomorphic functions on X, and $\mathcal{O}_X^\times$ the sheaf of nowhere-zero holomorphic functions on X (with group operation given by multiplication). Show that there is a short exact sequence of sheaves of abelian groups on X

$$0 \to \underline{\mathbb{Z}} \to \mathcal{O}_X \to \mathcal{O}_X^\times \to 0,$$

where the first map is the inclusion of the locally constant $\mathbb{Z}$-valued functions in the holomorphic functions, and the second map is given by $f \mapsto \exp(2\pi i f)$. This sequence is usually called the *exponential sequence*.

[Hint: To show that the second map is surjective on stalks at $p \in X$, one needs to construct logarithms of functions $g \in \mathcal{O}_X^\times(U)$ for U a neighbourhood of p, possibly after shrinking U. We may assume U is contractible, then, first choosing a logarithm of $g(p)$, we define

$$\log g(q) = \log g(p) + \int_{\gamma_q} \frac{dg}{g}$$

for $q \in U$, where γ_q is a path from p to q in U (this is well defined by Stokes' theorem).]

(b) Now assume $X = \mathbb{C}$. Show that the map $\mathcal{O}_X(U) \to \mathcal{O}_X^\times(U)$ is *not* surjective for $U = \mathbb{C} \setminus \{0\}$. Give a necessary and sufficient condition on an open set $V \subset \mathbb{C}$ for $\mathcal{O}_X(V) \to \mathcal{O}_X^\times(V)$ to be surjective.

(3) Let X be a variety and $Y \subset X$ a closed subvariety. Then we have a short exact sequence of sheaves on X

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to i_* \mathcal{O}_Y \to 0.$$

Here $\mathcal{I}_Y \subset \mathcal{O}_X$ is the ideal sheaf of regular functions on X vanishing on Y and $i \colon Y \to X$ is the inclusion.

(a) Check that the above sequence of sheaves is exact as follows: for $p \in X$ a point, let $U \subset X$ be an open affine subvariety such that $p \in U$, let $A = k[U]$ be the coordinate ring of U, $I = I(Y \cap U) \subset A$ the prime ideal corresponding to $Y \cap U$, and $\mathfrak{m} = I(p) \subset A$ the maximal ideal corresponding to p. Then the sequence of stalks at p is given by the localization of the exact sequence of A-modules

$$0 \to I \to A \to A/I \to 0$$

at the maximal ideal $\mathfrak{m}$.

(b) Suppose $X = \mathbb{P}^n$ and $Y = V(F) \subset X$ is the hypersurface defined by an irreducible homogeneous polynomial $F \in k[X_0, \ldots, X_n]$ of degree d. Show that we have an isomorphism of sheaves of $\mathcal{O}_{\mathbb{P}^n}$-modules

$$\mathcal{O}_{\mathbb{P}^n}(-d) \xrightarrow{\sim} \mathcal{I}_Y, \quad s \mapsto s \cdot F.$$

Now use the long exact sequence of cohomology for the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to i_* \mathcal{O}_Y \to 0$$

and the known cohomology groups of $\mathcal{O}_{\mathbb{P}^n}(m)$, $m \in \mathbb{Z}$ to show that

$$\dim_k H^i(Y, \mathcal{O}_Y) = \begin{cases} 1 & \text{if } i = 0, \\ \binom{d-1}{n} & \text{if } i = n-1 = \dim Y, \\ 0 & \text{otherwise.} \end{cases}$$

(4) In this question, we will discuss the general definition of Cech cohomology for sheaves of abelian groups on a topological space X. (See also Griffiths and Harris, p. 38–43 for further details and motivation, and Mumford and Oda, Ch. VII, for complete proofs.) For an open

covering $\mathcal{U} = \{U_i\}_{i \in I}$ (say for simplicity finite), as in class we define the associated Cech complex by

$$C^p(\mathcal{U}, \mathcal{F}) = \bigoplus_{i_0 < i_1 < \cdots < i_p} \mathcal{F}(U_{i_0 \cdots i_p})$$

(where we write $U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}$)) with differential $\delta \colon C^p \to C^{p+1}$ given by

$$(\delta s)_{i_0 \cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j s_{i_0 \cdots \widehat{i_j} \cdots i_{p+1}} |_{U_{i_0 \cdots i_{p+1}}}.$$

One checks that $\delta^2 = 0$, and defines

$$H^p(\mathcal{U}, \mathcal{F}) = \ker(\delta \colon C^p \to C^{p+1}) / \operatorname{im}(\delta \colon C^{p-1} \to C^p),$$

the pth cohomology group of the complex.

If $\mathcal{V} = \{V_j\}_{j \in J}$ is a refinement of $\mathcal{U}$, that is, there exists $f \colon J \to I$ such that $V_j \subset U_{f(j)}$ for all j, we can define a morphism of complexes

$$\theta \colon C^\bullet(\mathcal{U}, F) \to C^\bullet(\mathcal{V}, F)$$

via $\theta(s)_{j_0 \cdots j_p} = s_{f(j_0) \cdots f(j_p)}|_{V_{j_0 \cdots j_p}}$. (Here we extend the definition of $s_{i_0 \cdots i_p}$ from the case $i_0 < \ldots < i_p$ to the general case by requiring $s_{\sigma(i_0) \cdots \sigma(i_p)} = \operatorname{sgn}(\sigma) s_{i_0 \cdots i_p}$ for σ a permutation of $i_0, \ldots, i_p$.) Although this depends on the choice of $f \colon J \to I$, one shows that the induced maps on the cohomology groups of the complexes do not. Finally one defines the Cech cohomology group

$$H^p(X, \mathcal{F}) = \varinjlim_{\mathcal{U}} H^p(\mathcal{U}, \mathcal{F})$$

where the direct limit is over the set of all open coverings $\mathcal{U}$ of X with the partial ordering given by refinement.

The Leray theorem then asserts the following: if $\mathcal{U} = \{U_i\}_{i \in I}$ is an open covering such that $H^q(U_{i_0 \cdots i_p}, \mathcal{F}) = 0$ for all $p \geq 0$, $i_0, \ldots, i_p \in I$, and $q > 0$, then $H^p(\mathcal{U}, \mathcal{F}) = H^p(X, \mathcal{F})$ for all p. That is, the Cech cohomology groups are computed by the open covering $\mathcal{U}$. In the case of an algebraic variety and a coherent sheaf $\mathcal{F}$, this condition is satisfied for $\mathcal{U}$ a finite open covering by open affine sets. (This explains the definition of cohomology of coherent sheaves given in class.)

(a) Let $X = |\Sigma|$ be the topological realization of a finite simplicial complex Σ. For v a vertex of Σ, let $U_v = \mathrm{Star}(v) \subset X$ be the open set given by the union of the interiors of the simplices σ such that $v \in \sigma$. Show that the Cech complex for the sheaf $\underline{\mathbb{Z}}$ on X and the open covering $\mathcal{U} = \{U_v\}_{v \in V}$ is identified with the complex computing the simplicial cohomology of X. So $H^p(\mathcal{U}, \underline{\mathbb{Z}}) = H^p(X, \mathbb{Z})$, the (simplicial) cohomology of X with integral coefficients.

Moreover, if Σ' is a subdivision of Σ, and $\mathcal{U}'$ is the corresponding refinement of $\mathcal{U}$, show that the induced map $H^p(\mathcal{U}, \underline{\mathbb{Z}}) \to H^p(\mathcal{U}', \underline{\mathbb{Z}})$ corresponds to the identity map $H^p(X, \mathbb{Z}) \to H^p(X, \mathbb{Z})$. Since any open covering $\mathcal{V}$ of X is refined by the open covering associated to some subdivision of Σ, it follows that

$$H^p(X, \underline{\mathbb{Z}}) := \varinjlim_{\mathcal{V}} H^p(\mathcal{V}, \underline{\mathbb{Z}}) = H^p(X, \mathbb{Z}).$$

[If you have trouble, see Griffiths and Harris, p. 42–43.]

(b) Let X be a complex manifold. Consider the exponential sequence (see Q2) and the associated long exact sequence of cohomology

$$0 \to H^0(X, \mathbb{Z}) \to H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X^\times)$$

$$\xrightarrow{\delta} H^1(X, \underline{\mathbb{Z}}) \to H^1(X, \mathcal{O}_X) \to \cdots$$

Show that the connecting homomorphism

$$\delta \colon H^0(X, \mathcal{O}_X^\times) \to H^1(X, \underline{\mathbb{Z}}) = H^1(X, \mathbb{Z}) = \mathrm{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z})$$

is given by the winding number:

$$\delta(g) = \left(\gamma \mapsto \frac{1}{2\pi i} \int_\gamma \frac{dg}{g}\right).$$

[Note: The result of part (a) is valid for infinite simplicial complexes. This is needed for X non-compact. (If X is compact then $H^0(\mathcal{O}_X) = \mathbb{C}$ by the maximum principle, so $\delta = 0$ in this case.)]

(c) Let X be a complex manifold and $\mathrm{Pic}(X)$ the group of isomorphism classes of holomorphic line bundles on X, with group law the tensor product. Here, by a holomorphic line bundle we mean a morphism $\pi\colon L \to X$ of complex manifolds such that there is an open covering $\mathcal{U} = \{U_i\}_{i\in I}$ of X and *local trivializations*

$$\varphi_i\colon \pi^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{C}$$

over U_i such that the *transition maps*

$$\varphi_j \circ \varphi_i^{-1}\colon U_{ij} \times \mathbb{C} \xrightarrow{\sim} U_{ij} \times \mathbb{C}$$

are given by

$$\varphi_j \circ \varphi_i^{-1}(p, v) = (p, g_{ij}(p) \cdot v)$$

where $g_{ij} \in \mathcal{O}^\times(U_{ij})$ is a nowhere-zero holomorphic function on U_{ij}. Show that the assignment $L \mapsto [(g_{ij})]$ defines an isomorphism of abelian groups

$$\operatorname{Pic}(X) \xrightarrow{\sim} H^1(X, \mathcal{O}_X^\times) := \varinjlim_{\mathcal{U}} H^1(\mathcal{U}, \mathcal{O}_X^\times).$$

[Note: The same result holds for X a variety and $\operatorname{Pic}(X)$ the group of isomorphism classes of algebraic line bundles. However, note that $\mathcal{O}_X^\times$ is *not* a coherent sheaf (it is not even a sheaf of $\mathcal{O}_X$-modules), so the Cech cohomology group $H^1(X, \mathcal{O}_X^\times)$ is *not* computed by a single open affine covering $\mathcal{U}$ of X in general.]

(d) Let X be a compact complex manifold. Show that the long exact sequence of cohomology for the exponential sequence gives an exact sequence of abelian groups

$$0 \to H^1(X, \mathbb{Z}) \to H^1(X, \mathcal{O}_X) \to \operatorname{Pic} X$$

$$\xrightarrow{\delta} H^2(X, \mathbb{Z}) \to H^2(X, \mathcal{O}_X) \to \cdots$$

(e) Let X be a complex smooth projective variety, and X^{an} be the associated compact complex manifold. If E is an algebraic vector bundle over X, let E^{an} be the associated holomorphic vector bundle over X^{an}, and write $\mathcal{E}$, $\mathcal{E}^{\text{an}}$ for the sheaves of sections. Then $H^p(X, \mathcal{E}) = H^p(X^{\text{an}}, \mathcal{E}^{\text{an}})$. Moreover, the map $L \mapsto L^{\text{an}}$ gives an isomorphism $\operatorname{Pic}(X) \xrightarrow{\sim} \operatorname{Pic}(X^{\text{an}})$. (See Serre, Géométrie algébrique et géométrie analytique (1956).) So, we can use part (d) to compute $\operatorname{Pic}(X)$. As an example, show that $\operatorname{Pic}(\mathbb{P}^n) \simeq \mathbb{Z}$.

(5) Let X be a smooth projective curve. In class we described an isomorphism

$$\operatorname{Pic}(X) \xrightarrow{\sim} \operatorname{Cl}(X), \quad \mathcal{L} \mapsto (s),$$

where $0 \neq s \in \Gamma(U, \mathcal{L})$ is a non-zero section of $\mathcal{L}$ over an open set $U \subset X$, (s) is the divisor of zeroes and poles of s, and $\operatorname{Cl}(X)$ is the *divisor class group* of X, the abelian group of divisors D on X modulo the subgroup of principal divisors (f) for $f \in k(X)^\times$. The inverse is given by $D \mapsto \mathcal{O}_X(D)$, where $\mathcal{O}_X(D)$ is the (sheaf of sections of the) line bundle defined by

$$\Gamma(U, \mathcal{O}_X(D)) = \{f \in k(X)^\times \mid ((f) + D)|_U \geq 0\} \cup \{0\}$$

for $U \subset X$ open. [Note: An analogous statement holds for smooth varieties of dimension > 1 as well.]

Since $\deg(f) = 0$ for all $f \in k(X)^\times$, we have a group homomorphism

$$\deg\colon \operatorname{Cl}(X) \to \mathbb{Z}.$$

(a) Show that $\deg\colon \operatorname{Cl}(\mathbb{P}^1) \to \mathbb{Z}$ is an isomorphism.

(b) Suppose X has genus 1, and choose a basepoint $p_0 \in X$. Let $\operatorname{Cl}^0(X)$ be the kernel of $\deg\colon \operatorname{Cl}(X) \to \mathbb{Z}$. For $[D] \in \operatorname{Cl}^0(X)$, show using the Riemann-Roch formula that $\dim_k H^0(\mathcal{O}_X(D + p_0)) = 1$, and deduce that there is a unique $p \in X$ such that $[D] = [p - p_0] \in \operatorname{Cl}^0(X)$. So we have a bijection of sets

$$X \to \operatorname{Cl}^0(X), \quad p \mapsto [p - p_0].$$

In particular, X is naturally an abelian group with identity element p_0.

[Remark: In general, for $k = \mathbb{C}$, using Q4(d) and (e) one can show that $\operatorname{Cl}^0(X) \simeq \operatorname{Pic}^0 X \simeq H^1(X, \mathcal{O}_X)/H^1(X, \mathbb{Z})$. By Hodge theory, this quotient is a complex torus of dimension the genus g of X (that is, isomorphic to $\mathbb{C}^g / \bigoplus_{i=1}^{2g} \mathbb{Z}\lambda_i$ where $\lambda_1, \ldots, \lambda_{2g}$ is a basis of $\mathbb{C}^g$ as an $\mathbb{R}$-vector space).]

(6) Let X be an algebraic variety. For $\mathcal{E}$ a vector bundle on X of rank r, let $\det \mathcal{E} := \wedge^r \mathcal{E}$ be the top exterior power of $\mathcal{E}$. (So, if $\mathcal{E}$ has transition functions $g_{ij} \in \operatorname{GL}_r(\mathcal{O}_X(U_{ij}))$, then $\wedge^r \mathcal{E}$ is the line bundle

with transition functions $\det(g_{ij})$.) In particular, if X is smooth, the *canonical line bundle* ω_X of X is defined by

$$\omega_X = \det \Omega_X = \wedge^{\dim X} \Omega_X.$$

(a) Let $0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{E}'' \to 0$ be a short exact sequence of vector bundles on X. Show that

$$\det \mathcal{E} \simeq \det \mathcal{E}' \otimes \det \mathcal{E}''.$$

[Hint: For the block matrix $M = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ we have $\det M = \det A \cdot \det C$.]

(b) Recall the exact sequence of vector bundles on $\mathbb{P}^n$ (the *Euler sequence*)

$$0 \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus n+1} \to \mathcal{T}_{\mathbb{P}^n} \to 0$$

where $\mathcal{T}_{\mathbb{P}^n}$ is the tangent bundle of $\mathbb{P}^n$ and the first map is given by $1 \mapsto (X_0, \ldots, X_n)$. (See Griffiths and Harris, p. 408–409, or Hartshorne, Ch. II, Theorem 8.13, p. 176.) Dualizing, we have the exact sequence

$$0 \to \Omega_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

Deduce that $\omega_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-(n+1))$.

[Alternatively, one can compute that for the rational n-form $\omega = dx_1 \wedge \cdots \wedge dx_n$ on $\mathbb{P}^n$, where $x_i = X_i/X_0$, the divisor of zeroes and poles is given by $(\omega) = -(n+1)H$ where H is the hyperplane $(X_0 = 0) \subset \mathbb{P}^n$.]

(c) Let $n \geq 2$ and $Y \subset \mathbb{P}^n$ be the hypersurface defined by an irreducible homogeneous polynomial $F \in k[X_0, \ldots, X_n]$ of degree d. Assume Y is smooth. Show that there is an exact sequence of sheaves on Y

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d)|_Y \to \Omega_{\mathbb{P}^n}|_Y \to \Omega_Y \to 0$$

where the first map is given on $U_i = (X_i \neq 0) \simeq \mathbb{A}^n$ by $1/X_i^d \mapsto d(F/X_i^d)$, the differential of the equation of $Y \cap U_i \subset U_i$.

[Remark: The above exact sequence is dual to the exact sequence

$$0 \to \mathcal{T}_Y \to \mathcal{T}_{\mathbb{P}^n}|_Y \to \mathcal{N}_{Y/\mathbb{P}^n} \to 0$$

where $\mathcal{N}_{Y/\mathbb{P}^n}$ is the *normal bundle* of Y in $\mathbb{P}^n$.]

(d) Deduce that $\omega_Y \simeq (\omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(d))|_Y \simeq \mathcal{O}_{\mathbb{P}^n}(d-n-1)|_Y$.
[Remark: This is a special case of the *adjunction formula*

$$\omega_Y = (\omega_X \otimes \mathcal{O}_X(Y))|_Y$$

for X a smooth variety and $Y \subset X$ a smooth subvariety of codimension 1.]

(e) Using part (d), show that $\dim_k H^0(Y, \omega_Y) = \binom{d-1}{n}$. (Alternatively, this follows from Q3b and Serre duality.)
[Hint: Tensor the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n}(-d) \to \mathcal{O}_{\mathbb{P}^n} \to i_*\mathcal{O}_Y \to 0$$

by the line bundle $\mathcal{O}_{\mathbb{P}^n}(d-n-1)$ and consider the associated long exact sequence of cohomology.]