Math 797W Homework 3

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We work over an algebraically closed field k.

(1) Let $F_1, \ldots, F_m \in k[X_1, \ldots, X_n]$ be m non-constant homogeneous polynomials in n variables over an algebraically closed field k. Show that if $m < n$ then the system of equations

$$F_1 = F_2 = \cdots = F_m = 0$$

has a non-trivial solution $0 \neq (a_1, \ldots, a_n) \in k^n$.

[Remark: If each F_i has degree 1 then this is a basic result of linear algebra, valid for any field k.]

(2) Let $g \in k[x, y]$ be an irreducible polynomial. Let $X = V(z^3 - g) \subset \mathbb{A}^3_{x,y,z}$ and $Y = \mathbb{A}^2$. Let $f \colon X \to Y$ be the morphism given by $(x, y, z) \mapsto (x, y)$. Let $h \in k[X]$ be such that $Z = V(h) \subset X$ is irreducible. Find the equation of the image $f(Z) \subset Y$ in terms of h.

[Hint: Write $h = a_0 + a_1 z + a_2 z^2$ where $a_0, a_1, a_2 \in k[x, y]$. Review the proof of Krull's principal ideal theorem given in class and in Mumford's red book, pp. 41–43.]

(3) Let $X \subset \mathbb{P}^n$ be a closed subset, not necessarily irreducible. Let Y be a variety and $f \colon X \to Y$ a map of sets such that the restriction to each irreducible component of X is a morphism. Suppose $f(X)$ is irreducible and every fiber of f is irreducible of the same dimension r. Prove that X is irreducible and $\dim X = \dim f(X) + r$.

[Hint: If $f \colon X \to Y$ is a morphism of varieties such that $\overline{f(X)} = Y$, then for all $p \in Y$ a point and $Z \subset f^{-1}(p)$ an irreducible component we have $\dim Z \geq \dim X - \dim Y$, moreover there is an open set $U \subset Y$ such that $U \subset f(X)$ and for all $p \in U$ and $Z \subset f^{-1}(p)$ an irreducible component we have $\dim Z = \dim X - \dim Y$. Also, if $f\colon X \to Y$ is a morphism of algebraic varieties and X is projective then $f(X) \subset Y$ is closed. (We proved this for $k = \mathbb{C}$ using compactness for the analytic topology. The general case can be proved by a different method, cf. Mumford's red book, pp. 54–57.)]

(4) In this question, we will show that that every cubic surface $X \subset \mathbb{P}^3$ contains a line. (In fact, it is a classical result due to Cayley and Salmon that every smooth cubic surface contains exactly 27 lines.)

Let $G(r,n)$ denote the *Grassmannian* of subspaces of k^n of dimension r. Equivalently, the Grassmannian $G(r,n)$ is the set of projective linear subspaces of $\mathbb{P}^{n-1}$ of dimension $r-1$. The Grassmannian $G(r,n)$ is a projective variety of dimension $r(n-r)$.

Let $M \simeq \mathbb{P}^{19}$ be the projectivization of the k-vector space of homogeneous polynomials in $X_0, \ldots, X_3$ of degree 3. So a point $[F]$ of M corresponds to a cubic surface $X = V(F) \subset \mathbb{P}^3$ (not necessarily irreducible).

Now let $I \subset M \times G(2,4)$ be the closed subset defined by

$$I = \{(X, L) \mid L \subset X\}.$$

That is, I is the set of pairs (X, L) consisting of a cubic surface $X \subset \mathbb{P}^3$ and a line $L \subset \mathbb{P}^3$ such that $L \subset X$. Let $f\colon I \to M$, $(X,L) \mapsto X$ and $g\colon I \to G(2,4)$, $(X,L) \mapsto L$ be the projection maps.

(a) Show that every fiber of g is isomorphic to $\mathbb{P}^{15}$. Deduce using Q3 that I is irreducible and $\dim I = 19$.

(b) Let $X = V(X_1X_2X_3 - X_0^3) \subset \mathbb{P}^3$. Show that X contains exactly 3 lines.

(c) Show that f is surjective, that is, every cubic surface contains a line.

(5) Let $X = V(a_0X_0^4 + a_1X_0^3X_1 + \cdots + a_NX_3^4) \subset \mathbb{P}^3$ be a quartic surface. Show that there is an irreducible homogeneous polynomial G such that X contains a line iff $G(a_0, \ldots, a_N) = 0$.

(6) Let X be a curve (a variety of dimension 1) and $p \in X$ a smooth point. Let t be a generator of the maximal ideal $m_{X,p} \subset \mathcal{O}_{X,p}$, a *local parameter* at $p \in X$. Show that $\mathcal{O}_{X,p}$ is a *discrete valuation ring* (cf. Prof. Tevelev's 612 notes, Definition 10.3.1, p. 74) as follows:

(a) For $0 \neq f \in \mathcal{O}_{X,p}$, show that there exist unique $n \in \mathbb{Z}_{\geq 0}$ and $u \in \mathcal{O}_{X,p}$ a unit such that $f = u \cdot t^n$. Similarly, for $0 \neq g \in k(X)$, the field of fractions of $\mathcal{O}_{X,p}$, there exist unique $n \in \mathbb{Z}$ and $u \in \mathcal{O}_{X,p}$ a unit such that $g = u \cdot t^n$.

[Hint: We have $\bigcap_{n \geq 1} m_{X,p}^n = (0)$, either by the Krull intersection theorem or by Nakayama's lemma (using $m_{X,p}$ principal).]

(b) Given $0 \neq g \in k(X)$ as in (a), define $\nu(g) = n \in \mathbb{Z}$. (We say $\nu(g)$ is the *order of vanishing* of g at p.) Show that $\nu \colon k(X)^\times \to \mathbb{Z}$ satisfies $\nu(gh) = \nu(g) + \nu(h)$ and $\nu(g+h) \geq \min(\nu(g), \nu(h))$ (when $g + h \neq 0$) for all $g, h \in k(X)^\times$. Show that

$$\mathcal{O}_{X,p} = \{g \in k(X)^\times \mid \nu(g) \geq 0\} \cup \{0\}.$$

We say ν is a *discrete valuation* and $\mathcal{O}_{X,p}$ is the associated *discrete valuation ring.*

(c) Recall that we have a injective ring homomorphism $\mathcal{O}_{X,p} \subset \hat{\mathcal{O}}_{X,p}$ and an identification of $\hat{\mathcal{O}}_{X,p}$ with $k[[x]]$, the formal power series ring in the variable x, given by $x \mapsto t$. Show that the field of fractions of $k[[x]]$ is the field of *formal Laurent series*

$$k((x)) = \left\{ \sum_{n \geq m} a_n x^n \;\middle|\; m \in \mathbb{Z}, a_n \in k \right\}.$$

Show that the valuation ν is induced by the valuation

$$\hat{\nu} \colon k((x))^\times \to \mathbb{Z}$$

given by $\hat{\nu}(f) = m$ for $f = \sum_{n \geq m} a_n x^n$, $a_m \neq 0$.

(7) Let $f \in k[x_1]$ be a polynomial of degree $d \geq 1$ with distinct roots, and

$$V_1 = V(y_1^2 - f(x_1)) \subset \mathbb{A}^2_{x_1, y_1}.$$

Write $l = \lceil d/2 \rceil$, so that $2l = d + \delta$ where $\delta = 0$ if d is even and $\delta = 1$ if d is odd.

As in HW1Q14, consider the surface $X = X(n) = U_1 \cup U_2$ for $n = -l$, where $U_1 = \mathbb{A}^2_{x_1,y_1}$ and $U_2 = \mathbb{A}^2_{x_2,y_2}$, with glueing given by

$$\mathbb{A}^2_{x_1,y_1} \supset (x_1 \neq 0) \xrightarrow{\sim} (x_2 \neq 0) \subset \mathbb{A}^2_{x_2,y_2},$$

$$(x_1, y_1) \mapsto (x_1^{-1}, x_1^n y_1).$$

Let $p \colon X \to \mathbb{P}^1$ be the morphism given in charts by $\mathbb{A}^2_{x_i,y_i} \to \mathbb{A}^1_{x_i}$, $(x_i, y_i) \mapsto x_i$.

Let Y be the closure of V_1 in X. Thus $Y = V_1 \cup V_2$ where $V_2 = Y \cap U_2$.

(a) Find the equation of the second affine chart $V_2 \subset \mathbb{A}^2_{x_2,y_2}$ for Y.

(b) Show that Y is smooth (cf. HW2Q4(a)).

(c) Show that p restricts to a finite morphism $f \colon Y \to \mathbb{P}^1$.

[Here, we say a morphism $f \colon X \to Y$ of varieties is *finite* if there is an affine open covering $Y = \bigcup U_i$ such that for each i the open subvariety $f^{-1}U_i \subset X$ is affine and the induced morphism $f^{-1}U_i \to U_i$ is finite, that is, $k[f^{-1}U_i]$ is a finitely generated $k[U_i]$-module.]

(d) Show that $|f^{-1}(p)| \leq 2$ for all $p \in \mathbb{P}^1$, and $|f^{-1}(p)| = 1$ for exactly $2l$ points $p \in \mathbb{P}^1$.

(e) Now suppose $k = \mathbb{C}$. Show that X is compact. Show that X is a Riemann surface of genus $g = l - 1$.

[Hint: For the genus calculation, compare Miles Reid, Undergraduate algebraic geometry, pp. 50–51.]

(8) Let Y be the curve described in Q7. Show that the k-vector space $\Omega_Y(Y)$ of global differentials on Y has dimension $l - 1$.

[Hint: Compare the analogous calculation for a plane curve given in class.]