We work over an algebraically closed field $k$.

(1) Let $f : X \to Y$ be a morphism from a projective variety $X$ to an affine variety $Y$. Show that $f(X)$ is a point.

(2) Consider the morphism

$$f : \mathbb{P}^1 \to \mathbb{P}^3, \quad (X_0 : X_1) \mapsto (X_3^3 : X_0^2X_1 : X_0X_1^2 : X_1^3).$$

Let $X = f(\mathbb{P}^1) \subset \mathbb{P}^3$, the rational normal curve of degree 3.

(a) Show that $X = V(J)$ where

$$J := (Y_0Y_2 - Y_1^2, Y_1Y_3 - Y_2^2, Y_0Y_3 - Y_1Y_2).$$

(b) Prove that $J$ is the kernel of the ring homomorphism

$$k[Y_0, Y_1, Y_2, Y_3] \to k[X_0, X_1], \quad Y_0, Y_1, Y_2, Y_3 \mapsto X_3^3, X_0^2X_1, X_0X_1^2, X_1^3.$$ 

Deduce that $J$ is prime and hence $J = I(X)$.

(3) Let

$$f : \mathbb{P}^2 \to X \subset \mathbb{P}^5$$

$$(X_0 : X_1 : X_2) \mapsto (X_0^3 : X_1^3 : X_2^3 : X_0X_1 : X_0X_2 : X_1X_2)$$

be the Veronese surface in $\mathbb{P}^5$ (the second Veronese embedding of $\mathbb{P}^2$).

(a) Show that $f$ induces a bijective correspondence between hyperplanes $H$ in $\mathbb{P}^5$ and curves $Y = V(F) \subset \mathbb{P}^2$, where $F \in k[X_0, X_1, X_2]$ is a homogeneous polynomial of degree 2, via $H \mapsto f^{-1}H$. (Note: $F$ need not be irreducible.)
(b) Deduce that, for \( Y \subset \mathbb{P}^2 \) as in (a), the open subset \( U := \mathbb{P}^2 \setminus Y \subset \mathbb{P}^2 \) is isomorphic to an affine variety.

(c) More generally, show that if \( X \subset \mathbb{P}^n \) is a projective variety, and \( Y = V(F) \subset \mathbb{P}^n \) where \( F \in k[X_0, \ldots, X_n] \) is a homogeneous polynomial of some degree \( d \), then the open subset \( U := X \setminus Y \subset X \) is isomorphic to an affine variety.

(4) Assume \( \text{char}(k) \neq 2 \). Let \( X = V(y^2 - f(x)) \subset \mathbb{A}^2_{x,y} \) where \( f \in k[x] \) is a polynomial of degree \( d \geq 1 \) with distinct roots.

(a) Show that \( X \) is smooth.

(b) Let \( \overline{X} \) denote the closure of \( X \) in \( \mathbb{P}^2 \). Compute the homogeneous equation of \( \overline{X} \)

(c) Determine the set \( \overline{X} \setminus X \) for \( d \geq 3 \).

(d) Show that \( \overline{X} \) is not smooth if \( d \geq 4 \).

(5) Let \( F \in k[X_0, \ldots, X_n] \) be an irreducible homogeneous polynomial of degree \( d \) in \( n + 1 \) variables, for some \( n \geq 2 \), and \( X = V(F) \subset \mathbb{P}^n \) the associated projective hypersurface. Assume that \( \text{char}(k) \) does not divide \( d \).

(a) Prove Euler’s formula

\[
\sum_{i=0}^{n} X_i \cdot \frac{\partial F}{\partial X_i} = d \cdot F.
\]

(b) Show that the singular locus of \( X \) is given by

\[
\text{Sing}(X) = V\left( \frac{\partial F}{\partial X_0}, \ldots, \frac{\partial F}{\partial X_n} \right) \subset \mathbb{P}^n
\]

(c) Show that if \( F = X_0^d + X_1^d + \cdots + X_n^d \) then \( X \) is smooth.

(d) Suppose \( d = 2 \). Then, by the classification of quadratic forms, after a linear change of homogeneous coordinates on \( \mathbb{P}^n \) we may assume that \( F = X_0^2 + X_1^2 + \cdots + X_m^2 \), where \( m \leq n \). (Here \( m + 1 \) is the rank of the quadratic form \( F \).)

i. Show that \( X_0^2 + \cdots + X_m^2 \) is irreducible iff \( m \geq 2 \).
ii. Show that $X$ is smooth iff $m = n$ and identify the singular locus for $m < n$.

iii. Deduce that two smooth quadric hypersurfaces of the same dimension are isomorphic. In particular, for $X$ a smooth quadric hypersurface in $\mathbb{P}^n$, if $n = 2$ then $X \cong \mathbb{P}^1$ and if $n = 3$ then $X \cong \mathbb{P}^1 \times \mathbb{P}^1$ (why?).

(e) Assume $\text{char}(k) = 0$ and $d > 2$. Show that if $F = X_0^{d-1}X_1 + X_1^{d-1}X_2 + \cdots + X_n^{d-1}X_0$ then $X$ is smooth.

(6) Let $X \subset \mathbb{P}^4$ be a projective hypersurface of degree $d > 1$. Suppose that $X$ contains a plane $\Pi \subset \mathbb{P}^4$ (a projective linear subspace of dimension 2). Show that $X$ is necessarily singular.

(7) Let $\pi: \widetilde{\mathbb{A}}^n \to \mathbb{A}^n$ denote the blowup of the origin in $\mathbb{A}^n$. That is,

$$\widetilde{\mathbb{A}}^n = V(\{x_iX_j - x_jX_i \mid 1 \leq i < j \leq n\}) \subset \mathbb{A}^n_{x_1, \ldots, x_n} \times \mathbb{P}^{n-1}_{(X_1: \cdots : X_n)},$$

and the morphism $\pi: \widetilde{\mathbb{A}}^n \to \mathbb{A}^n$ is the restriction of the first projection $\mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$.

(a) Show that $\widetilde{\mathbb{A}}^n$ is covered by $n$ affine open subsets, each isomorphic to $\mathbb{A}^n$, and describe the restriction of $\pi$ to each chart explicitly.

(b) Show that $E := \pi^{-1}(0)$ is isomorphic to $\mathbb{P}^{n-1}$ and that $\widetilde{\mathbb{A}}^n \setminus E \to \mathbb{A}^n \setminus \{0\}$ is an isomorphism. What is the defining equation of $E$ in each chart?

(c) Now suppose $X \subset \mathbb{A}^n$ is an affine variety and $p \in X$ is a point. We can define the blowup $\pi_X: \tilde{X} \to X$ of $X$ at $p$ as follows (we do not assume $X$ is smooth at $p$ here). We may assume (applying a translation) that $p = 0 \in \mathbb{A}^n$. Let

$$\tilde{X} := \overline{\pi^{-1}(X \setminus \{0\})} \subset \widetilde{\mathbb{A}}^n,$$

the closure of the inverse image of $X \setminus \{0\}$, and let $\pi_X: \tilde{X} \to X$ be the restriction of $\pi$. Then, writing $F := \tilde{X} \cap E$, we have $F = \pi_X^{-1}(p)$ and $\pi_X$ induces an isomorphism $\tilde{X} \setminus F \sim X \setminus \{p\}$.

(The closed subvariety $\tilde{X} \subset \widetilde{\mathbb{A}}^n$ is sometimes called the strict transform of $X$ in $\widetilde{\mathbb{A}}^n$.)

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Show that if $k = \mathbb{C}$ then the morphism $\pi_X$ is proper for the analytic topology. That is, if $K \subset X \subset \mathbb{C}^n$ is compact then $\pi_X^{-1}K \subset \tilde{X}$ is compact.

(8) Let $X$ be a variety over $k = \mathbb{C}$, and $S \subset X$ the set of singularities of $X$. A resolution of singularities is a morphism $f : Y \to X$ such that $Y$ is a smooth variety, $f$ restricts to an isomorphism $Y \setminus f^{-1}(S) \cong X \setminus S$, and $f$ is proper for the analytic topology. (For the last condition, see Q7. It rules out for example taking $Y = X \setminus S$ and $f$ the inclusion.)

(a) Let $X = V(x^2 + y^2 + z^2) \subset \mathbb{A}^3_{x,y,z}$. Show that the blowup $\pi_X : \tilde{X} \to X$ of $0 \in X$ is a resolution of singularities, and that $F = \pi_X^{-1}(0)$ is isomorphic to $\mathbb{P}^1$.

[Hint: With notation as in Q7(c), what is the homogeneous equation of $F \subset E = \pi^{-1}(0) \cong \mathbb{P}^2$?]

(b) (Optional) Let $X = V(xy - z^n) \subset \mathbb{A}^3_{x,y,z}$, where $n \in \mathbb{N}$, $n \geq 2$. Show that there exists a resolution of singularities $f : Y \to X$ such that $f^{-1}(0)$ is a chain of $n - 1$ copies of $\mathbb{P}^1$. That is, $f^{-1}(0) = C_1 \cup \cdots \cup C_{n-1}$ where each $C_i$ is isomorphic to $\mathbb{P}^1$, $C_i$ and $C_{i+1}$ meet transversely at a single point for each $1 \leq i < n - 1$, and $C_i \cap C_j = \emptyset$ for $|i - j| > 1$.

[Hint: Blow up the singular point and use induction.]

(9) We say a polynomial $F \in k[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$ is bihomogeneous of degree $(d, e)$ if it is homogeneous of degree $d$ in the $X$ variables and homogeneous of degree $e$ in the $Y$ variables, that is,

$$F = \sum_{i_0 + \cdots + i_n = d, j_0 + \cdots + j_m = e} a_{i_0 \ldots i_n j_0 \ldots j_m} X_0^{i_0} \cdots X_n^{i_n} Y_0^{j_0} \cdots Y_m^{j_m}$$

Similarly to the case of $\mathbb{P}^n$, closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$ are defined by ideals generated by bihomogeneous polynomials.

Now consider $\mathbb{P}^1 \times \mathbb{P}^1$ and $F \in k[X_0, X_1, Y_0, Y_1]$ an irreducible bihomogeneous polynomial of degree $(d, e)$.

(a) Consider the projective variety $X := V(F) \subset \mathbb{P}^1 \times \mathbb{P}^1$. Show that for any $p \in \mathbb{P}^1$ the intersection $X \cap \{p\} \times \mathbb{P}^1$ consists of $e$ points counting multiplicities (unless $d = 1, e = 0$, and $X = \{p\} \times \mathbb{P}^1$) and
similarly $X \cap \mathbb{P}^1 \times \{p\}$ consists of $d$ points counting multiplicities (unless $d = 0$, $e = 1$, and $X = \mathbb{P}^1 \times \{p\}$). [This is similar to HW1Q11 so you need only explain briefly here.]

(b) Consider the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$:

$$f: \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow Y \subset \mathbb{P}^3,$$

$$((X_0: X_1), (Y_0: Y_1)) \mapsto (X_0Y_0: X_1Y_0: X_0Y_1: X_1Y_1).$$

Show that $f(X)$ is equal to the intersection of $Y$ with a hypersurface $V(G) \subset \mathbb{P}^3$ defined by some homogeneous polynomial $G \in k[Z_{00}, Z_{10}, Z_{01}, Z_{11}]$ iff $d = e$.

(10) Let $\pi: \tilde{\mathbb{P}}^2 \rightarrow \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at the point $p = (1: 0: 0)$. Explicitly, we can identify $\tilde{\mathbb{P}}^2$ with

$$X = V(X_1Y_2 - X_2Y_1) \subset \mathbb{P}^2_{(X_0:X_1:X_2)} \times \mathbb{P}^1_{(Y_1;Y_2)}$$

so that the morphism $\pi: \tilde{\mathbb{P}}^2 \rightarrow \mathbb{P}^2$ is given by the restriction of the first projection $\mathbb{P}^2 \times \mathbb{P}^1 \rightarrow \mathbb{P}^2$. Then, under the Segre embedding

$$f: \mathbb{P}^2 \times \mathbb{P}^1 \rightarrow X \subset \mathbb{P}^5$$

$$((X_0 : X_1 : X_2), (Y_1 : Y_2)) \mapsto (X_0Y_1 : X_1Y_1 : X_0Y_2 : X_1Y_2 : X_2Y_2)$$

we have $Z := f(X) = Y \cap H$ where $H \subset \mathbb{P}^5$ is the hyperplane $H = V(Z_{12} - Z_{21})$. Thus, identifying $H = \mathbb{P}^4_{(Z_{01};Z_{02};Z_{12};Z_{22})}$, we have an embedding $Z \subset \mathbb{P}^4$. The projective variety $Z \subset \mathbb{P}^4$ is called the cubic scroll.

(a) Let $E = \pi^{-1}(p) \subset X$. Let $L = V(X_0) \subset \mathbb{P}^2$ and $L' = \pi^{-1}(L) \subset X$ (note that $p \notin L$). Show that $f(E)$ is a line in $\mathbb{P}^4$ and $f(L')$ is a smooth conic in a plane in $\mathbb{P}^4$ (where the line and the plane in $\mathbb{P}^4$ are disjoint).

(b) For $M$ a line in $\mathbb{P}^2$ passing through $p$, let $M' := \pi^{-1}(M \setminus \{p\})$ be the strict transform of $M$. Equivalently, we have the morphism $g: X \rightarrow \mathbb{P}^1$ given by the restriction of the second projection $\mathbb{P}^2 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$, and as $M$ varies the curves $M' \subset X$ are precisely the fibers of the morphism $g$. Show that $M'$ maps to the line in $\mathbb{P}^4$ connecting the two points $f(E \cap M')$ and $f(L' \cap M')$. 

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(c) Deduce that there is an isomorphism $\varphi: E \overset{\sim}{\longrightarrow} L'$ such that $Z = f(X)$ is the disjoint union of the lines $L_p$ connecting the points $p$ and $\varphi(p)$ for $p \in E \cong \mathbb{P}^1$.

(d) (Optional) Show that the degree of $Z \subset \mathbb{P}^4$ equals 3.

[Hint: Recall that for $X \subset \mathbb{P}^n$ a projective variety of dimension $d$, the degree of $X$ equals the number of points in the intersection $X \cap H_1 \cap \cdots \cap H_d$ where $H_1, \ldots, H_d \subset \mathbb{P}^n$ are general hyperplanes (assuming $\text{char}(k) = 0$). More generally, whenever $X \cap H_1 \cap \cdots \cap H_d$ is a finite set, the degree of $X$ is equal to the number of points counted with multiplicities. One approach in our case: show that $Z \cap V(Z_{02}) = L' + V(Y_2)$ and $Z \cap V(Z_{11}) = E + 2 \cdot V(Y_1)$, so that the intersection $Z \cap V(Z_{02}) \cap V(Z_{11})$ consists of $1 + 2 = 3$ points counting multiplicities. Another approach (assuming $k = \mathbb{C}$): compute the intersection number in $H_*(\mathbb{P}^2 \times \mathbb{P}^1, \mathbb{Z})$ using the Künneth formula.]

(11) Consider the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$:

$$f: \mathbb{P}^1 \times \mathbb{P}^1 \overset{\sim}{\longrightarrow} X \subset \mathbb{P}^3,$$

$$((X_0 : X_1), (Y_0 : Y_1)) \mapsto (X_0 Y_0 : X_1 Y_0 : X_0 Y_1 : X_1 Y_1).$$

Here the image $X$ of $f$ is the quadric surface $V(Z_{00}Z_{11} - Z_{10}Z_{01}) \subset \mathbb{P}^3$.

Let $p = (0 : 0 : 0 : 1) \in X$ and consider the morphism

$$g: X \setminus \{p\} \rightarrow \mathbb{P}^2, \quad (Z_{00} : Z_{10} : Z_{01} : Z_{11}) \mapsto (Z_{00} : Z_{10} : Z_{01}).$$

Let $\pi_X: \hat{X} \rightarrow X$ be the blowup of $X$ at the point $p$ and write $E = \pi_X^{-1}(p)$.

Show that the composition $g \circ \pi_X: \hat{X} \setminus E \rightarrow \mathbb{P}^2$ extends to a morphism $h: \hat{X} \rightarrow \mathbb{P}^2$. Show that there is an open set $U \subset \mathbb{P}^2$ such that $h$ restricts to an isomorphism $h^{-1}(U) \overset{\sim}{\longrightarrow} U$, find the largest such open set $U$, and determine the fibers $h^{-1}(q)$ of $h$ for $q \notin U$.

[Hint: $\hat{X} \subset \overset{\sim}{\mathbb{P}}^3 \subset \mathbb{P}^3 \times \mathbb{P}^2$, cf. Q10.]