# Math 797W Homework 2

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We work over an algebraically closed field $k$.

(1) Let $f\colon X \to Y$ be a morphism from a projective variety $X$ to an affine variety $Y$. Show that $f(X)$ is a point.

(2) Consider the morphism

$$f\colon \mathbb{P}^1 \to \mathbb{P}^3, \quad (X_0 : X_1) \mapsto (X_0^3 : X_0^2X_1 : X_0X_1^2 : X_1^3).$$

Let $X = f(\mathbb{P}^1) \subset \mathbb{P}^3$, the *rational normal curve* of degree 3.

(a) Show that $X = V(J)$ where

$$J := (Y_0Y_2 - Y_1^2, Y_1Y_3 - Y_2^2, Y_0Y_3 - Y_1Y_2).$$

(b) Prove that $J$ is the kernel of the ring homomorphism

$$k[Y_0, Y_1, Y_2, Y_3] \to k[X_0, X_1], \quad Y_0, Y_1, Y_2, Y_3 \mapsto X_0^3, X_0^2X_1, X_0X_1^2, X_1^3.$$

Deduce that $J$ is prime and hence $J = I(X)$.

(3) Let

$$f\colon \mathbb{P}^2 \xrightarrow{\sim} X \subset \mathbb{P}^5$$
$$(X_0 : X_1 : X_2) \mapsto (X_0^2 : X_1^2 : X_2^2 : X_0X_1 : X_0X_2 : X_1X_2)$$

be the Veronese surface in $\mathbb{P}^5$ (the second Veronese embedding of $\mathbb{P}^2$).

(a) Show that $f$ induces a bijective correspondence between hyperplanes $H$ in $\mathbb{P}^5$ and curves $Y = V(F) \subset \mathbb{P}^2$, where $F \in k[X_0, X_1, X_2]$ is a homogeneous polynomial of degree 2, via $H \mapsto f^{-1}H$. (Note: $F$ need not be irreducible.)

(b) Deduce that, for $Y \subset \mathbb{P}^2$ as in (a), the open subset $U := \mathbb{P}^2 \setminus Y \subset \mathbb{P}^2$ is isomorphic to an affine variety.

(c) More generally, show that if $X \subset \mathbb{P}^n$ is a projective variety, and $Y = V(F) \subset \mathbb{P}^n$ where $F \in k[X_0, \ldots, X_n]$ is a homogeneous polynomial of some degree $d$, then the open subset $U := X \setminus Y \subset X$ is isomorphic to an affine variety.

(4) Assume $\mathrm{char}(k) \neq 2$. Let $X = V(y^2 - f(x)) \subset \mathbb{A}^2_{x,y}$ where $f \in k[x]$ is a polynomial of degree $d \geq 1$ with distinct roots.

(a) Show that $X$ is smooth.

(b) Let $\overline{X}$ denote the closure of $X$ in $\mathbb{P}^2$. Compute the homogeneous equation of $\overline{X}$

(c) Determine the set $\overline{X} \setminus X$ for $d \geq 3$.

(d) Show that $\overline{X}$ is not smooth if $d \geq 4$.

(5) Let $F \in k[X_0, \ldots, X_n]$ be an irreducible homogeneous polynomial of degree $d$ in $n + 1$ variables, for some $n \geq 2$, and $X = V(F) \subset \mathbb{P}^n$ the associated projective hypersurface. Assume that $\mathrm{char}(k)$ does not divide $d$.

(a) Prove Euler's formula

$$\sum_{i=0}^{n} X_i \cdot \frac{\partial F}{\partial X_i} = d \cdot F.$$

(b) Show that the singular locus of $X$ is given by

$$\mathrm{Sing}(X) = V\left(\frac{\partial F}{\partial X_0}, \cdots, \frac{\partial F}{\partial X_n}\right) \subset \mathbb{P}^n$$

(c) Show that if $F = X_0^d + X_1^d + \cdots + X_n^d$ then $X$ is smooth.

(d) Suppose $d = 2$. Then, by the classification of quadratic forms, after a linear change of homogeneous coordinates on $\mathbb{P}^n$ we may assume that $F = X_0^2 + X_1^2 + \cdots + X_m^2$, where $m \leq n$. (Here $m+1$ is the *rank* of the quadratic form $F$.)

i. Show that $X_0^2 + \cdots + X_m^2$ is irreducible iff $m \geq 2$.

ii. Show that $X$ is smooth iff $m = n$ and identify the singular locus for $m < n$.

iii. Deduce that two smooth quadric hypersurfaces of the same dimension are isomorphic. In particular, for $X$ a smooth quadric hypersurface in $\mathbb{P}^n$, if $n = 2$ then $X \simeq \mathbb{P}^1$ and if $n = 3$ then $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ (why?).

(e) Assume $\operatorname{char}(k) = 0$ and $d > 2$. Show that if $F = X_0^{d-1}X_1 + X_1^{d-1}X_2 + \cdots + X_n^{d-1}X_0$ then $X$ is smooth.

(6) Let $X \subset \mathbb{P}^4$ be a projective hypersurface of degree $d > 1$. Suppose that $X$ contains a plane $\Pi \subset \mathbb{P}^4$ (a projective linear subspace of dimension 2). Show that $X$ is necessarily singular.

(7) Let $\pi\colon \widetilde{\mathbb{A}^n} \to \mathbb{A}^n$ denote the blowup of the origin in $\mathbb{A}^n$. That is,

$$\widetilde{\mathbb{A}^n} = V(\{x_iX_j - x_jX_i \mid 1 \le i < j \le n\}) \subset \mathbb{A}^n_{x_1,\dots,x_n} \times \mathbb{P}^{n-1}_{(X_1:\cdots:X_n)},$$

and the morphism $\pi\colon \widetilde{\mathbb{A}^n} \to \mathbb{A}^n$ is the restriction of the first projection $\mathbb{A}^n \times \mathbb{P}^{n-1} \to \mathbb{A}^n$.

(a) Show that $\widetilde{\mathbb{A}^n}$ is covered by $n$ affine open subsets, each isomorphic to $\mathbb{A}^n$, and describe the restriction of $\pi$ to each chart explicitly.

(b) Show that $E := \pi^{-1}(0)$ is isomorphic to $\mathbb{P}^{n-1}$ and that $\widetilde{\mathbb{A}^n} \setminus E \to \mathbb{A}^n \setminus \{0\}$ is an isomorphism. What is the defining equation of $E$ in each chart?

(c) Now suppose $X \subset \mathbb{A}^n$ is an affine variety and $p \in X$ is a point. We can define the blowup $\pi_X\colon \tilde{X} \to X$ of $X$ at $p$ as follows (we do *not* assume $X$ is smooth at $p$ here). We may assume (applying a translation) that $p = 0 \in \mathbb{A}^n$. Let

$$\tilde{X} := \overline{\pi^{-1}(X \setminus \{0\})} \subset \widetilde{\mathbb{A}^n},$$

the closure of the inverse image of $X \setminus \{0\}$, and let $\pi_X\colon \tilde{X} \to X$ be the restriction of $\pi$. Then, writing $F := \tilde{X} \cap E$, we have $F = \pi_X^{-1}(p)$ and $\pi_X$ induces an isomorphism $\tilde{X} \setminus F \xrightarrow{\sim} X \setminus \{p\}$. (The closed subvariety $\tilde{X} \subset \widetilde{\mathbb{A}^n}$ is sometimes called the *strict transform* of $X$ in $\widetilde{\mathbb{A}^n}$.)

Show that if $k = \mathbb{C}$ then the morphism $\pi_X$ is proper for the analytic topology. That is, if $K \subset X \subset \mathbb{C}^n$ is compact then $\pi_X^{-1}K \subset \tilde{X}$ is compact.

(8) Let $X$ be a variety over $k = \mathbb{C}$, and $S \subset X$ the set of singularities of $X$. A *resolution of singularities* is a morphism $f\colon Y \to X$ such that $Y$ is a smooth variety, $f$ restricts to an isomorphism $Y \setminus f^{-1}(S) \xrightarrow{\sim} X \setminus S$, and $f$ is proper for the analytic topology. (For the last condition, see Q7. It rules out for example taking $Y = X \setminus S$ and $f$ the inclusion.)

(a) Let $X = V(x^2+y^2+z^2) \subset \mathbb{A}^3_{x,y,z}$. Show that the blowup $\pi_X\colon \tilde{X} \to X$ of $0 \in X$ is a resolution of singularities, and that $F = \pi_X^{-1}(0)$ is isomorphic to $\mathbb{P}^1$.

[Hint: With notation as in Q7(c), what is the homogeneous equation of $F \subset E = \pi^{-1}(0) \simeq \mathbb{P}^2$?]

(b) (Optional) Let $X = V(xy - z^n) \subset \mathbb{A}^3_{x,y,z}$, where $n \in \mathbb{N}$, $n \geq 2$. Show that there exists a resolution of singularities $f\colon Y \to X$ such that $f^{-1}(0)$ is a chain of $n-1$ copies of $\mathbb{P}^1$. That is, $f^{-1}(0) = C_1 \cup \cdots \cup C_{n-1}$ where each $C_i$ is isomorphic to $\mathbb{P}^1$, $C_i$ and $C_{i+1}$ meet transversely at a single point for each $1 \leq i < n-1$, and $C_i \cap C_j = \emptyset$ for $|i-j| > 1$.

[Hint: Blowup the singular point and use induction.]

(9) We say a polynomial $F \in k[X_0, \ldots, X_n, Y_0, \ldots, Y_m]$ is *bihomogeneous* of degree $(d,e)$ if it is homogeneous of degree $d$ in the $X$ variables and homogeneous of degree $e$ in the $Y$ variables, that is,

$$F = \sum_{\substack{i_0+\cdots+i_n=d \\ j_0+\cdots+j_m=e}} a_{i_0\cdots i_n j_0 \cdots j_m} X_0^{i_0} \cdots X_n^{i_n} Y_0^{j_0} \cdots Y_m^{j_m}$$

Similarly to the case of $\mathbb{P}^n$, closed subsets of $\mathbb{P}^n \times \mathbb{P}^m$ are defined by ideals generated by bihomogeneous polynomials.

Now consider $\mathbb{P}^1 \times \mathbb{P}^1$ and $F \in k[X_0, X_1, Y_0, Y_1]$ an irreducible bihomogeneous polynomial of degree $(d,e)$.

(a) Consider the projective variety $X := V(F) \subset \mathbb{P}^1 \times \mathbb{P}^1$. Show that for any $p \in \mathbb{P}^1$ the intersection $X \cap \{p\} \times \mathbb{P}^1$ consists of $e$ points counting multiplicities (unless $d = 1$, $e = 0$, and $X = \{p\} \times \mathbb{P}^1$) and similarly $X \cap \mathbb{P}^1 \times \{p\}$ consists of $d$ points counting multiplicities (unless $d = 0$, $e = 1$, and $X = \mathbb{P}^1 \times \{p\}$). [This is similar to HW1Q11 so you need only explain briefly here.]

(b) Consider the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$:

$$f\colon \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sim} Y \subset \mathbb{P}^3,$$

$$((X_0\colon X_1), (Y_0\colon Y_1)) \mapsto (X_0Y_0\colon X_1Y_0\colon X_0Y_1\colon X_1Y_1).$$

Show that $f(X)$ is equal to the intersection of $Y$ with a hypersurface $V(G) \subset \mathbb{P}^3$ defined by some homogeneous polynomial $G \in k[Z_{00}, Z_{10}, Z_{01}, Z_{11}]$ iff $d = e$.

(10) Let $\pi\colon \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ be the blowup of $\mathbb{P}^2$ at the point $p = (1:0:0)$. Explicitly, we can identify $\widetilde{\mathbb{P}^2}$ with

$$X = V(X_1Y_2 - X_2Y_1) \subset \mathbb{P}^2_{(X_0:X_1:X_2)} \times \mathbb{P}^1_{(Y_1:Y_2)}$$

so that the morphism $\pi\colon \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ is given by the restriction of the first projection $\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^2$. Then, under the Segre embedding

$$f\colon \mathbb{P}^2 \times \mathbb{P}^1 \xrightarrow{\sim} Y \subset \mathbb{P}^5$$

$$((X_0 : X_1 : X_2), (Y_1 : Y_2)) \mapsto (X_0Y_1 : X_1Y_1 : X_2Y_1 : X_0Y_2 : X_1Y_2 : X_2Y_2)$$

we have $Z := f(X) = Y \cap H$ where $H \subset \mathbb{P}^5$ is the hyperplane $H = V(Z_{12} - Z_{21})$. Thus, identifying $H = \mathbb{P}^4_{(Z_{01}:Z_{11}:Z_{02}:Z_{12}:Z_{22})}$, we have an embedding $Z \subset \mathbb{P}^4$. The projective variety $Z \subset \mathbb{P}^4$ is called the *cubic scroll*.

(a) Let $E = \pi^{-1}(p) \subset X$. Let $L = V(X_0) \subset \mathbb{P}^2$ and $L' = \pi^{-1}(L) \subset X$ (note that $p \notin L$). Show that $f(E)$ is a line in $\mathbb{P}^4$ and $f(L')$ is a smooth conic in a plane in $\mathbb{P}^4$ (where the line and the plane in $\mathbb{P}^4$ are disjoint).

(b) For $M$ a line in $\mathbb{P}^2$ passing through $p$, let $M' := \overline{\pi^{-1}(M \setminus \{p\})}$ be the strict transform of $M$. Equivalently, we have the morphism $g\colon X \to \mathbb{P}^1$ given by the restriction of the second projection $\mathbb{P}^2 \times \mathbb{P}^1 \to \mathbb{P}^1$, and as $M$ varies the curves $M' \subset X$ are precisely the fibers of the morphism $g$. Show that $M'$ maps to the line in $\mathbb{P}^4$ connecting the two points $f(E \cap M')$ and $f(L' \cap M')$.

(c) Deduce that there is an isomorphism $\varphi\colon E \xrightarrow{\sim} L'$ such that $Z = f(X)$ is the disjoint union of the lines $L_p$ connecting the points $p$ and $\varphi(p)$ for $p \in E \simeq \mathbb{P}^1$.

(d) (Optional) Show that the degree of $Z \subset \mathbb{P}^4$ equals 3.

[Hint: Recall that for $X \subset \mathbb{P}^n$ a projective variety of dimension $d$, the degree of $X$ equals the number of points in the intersection $X \cap H_1 \cap \cdots \cap H_d$ where $H_1, \ldots, H_d \subset \mathbb{P}^n$ are general hyperplanes (assuming $\mathrm{char}(k) = 0$). More generally, whenever $X \cap H_1 \cap \cdots \cap H_d$ is a finite set, the degree of $X$ is equal to the number of points counted with multiplicities. One approach in our case: show that $Z \cap V(Z_{02}) = L' + V(Y_2)$ and $Z \cap V(Z_{11}) = E + 2 \cdot V(Y_1)$, so that the intersection $Z \cap V(Z_{02}) \cap V(Z_{11})$ consists of $1 + 2 = 3$ points counting multiplicities. Another approach (assuming $k = \mathbb{C}$): compute the intersection number in $H_*(\mathbb{P}^2 \times \mathbb{P}^1, \mathbb{Z})$ using the Künneth formula.]

(11) Consider the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$:

$$f\colon \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sim} X \subset \mathbb{P}^3,$$

$$((X_0 : X_1), (Y_0 : Y_1)) \mapsto (X_0Y_0 : X_1Y_0 : X_0Y_1 : X_1Y_1).$$

Here the image $X$ of $f$ is the quadric surface $V(Z_{00}Z_{11} - Z_{10}Z_{01}) \subset \mathbb{P}^3$. Let $p = (0 : 0 : 0 : 1) \in X$ and consider the morphism

$$g\colon X \setminus \{p\} \to \mathbb{P}^2, \quad (Z_{00} : Z_{10} : Z_{01} : Z_{11}) \mapsto (Z_{00} : Z_{10} : Z_{01}).$$

Let $\pi_X\colon \tilde{X} \to X$ be the blowup of $X$ at the point $p$ and write $E = \pi_X^{-1}(p)$.

Show that the composition $g \circ \pi_X\colon \tilde{X} \setminus E \to \mathbb{P}^2$ extends to a morphism $h\colon \tilde{X} \to \mathbb{P}^2$. Show that there is an open set $U \subset \mathbb{P}^2$ such that $h$ restricts to an isomorphism $h^{-1}(U) \xrightarrow{\sim} U$, find the largest such open set $U$, and determine the fibers $h^{-1}(q)$ of $h$ for $q \notin U$.

[Hint: $\tilde{X} \subset \widetilde{\mathbb{P}^3} \subset \mathbb{P}^3 \times \mathbb{P}^2$, cf. Q10.]