Math 797W Homework 1

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We work over an algebraically closed field k (unless explicitly stated otherwise). Questions 1 and 2 are preliminary and will not be graded.

(1) We say a topological space X is *irreducible* if there does not exist a decomposition $X = X_1 \cup X_2$ where $X_1, X_2 \subsetneq X$ are closed subsets. Prove the following statements.

(a) X is irreducible iff for all non-empty open sets $U_1, U_2 \subset X$ the intersection $U_1 \cap U_2$ is non-empty.

(b) If X is irreducible and $U \subset X$ is a non-empty open set, then U is dense (that is, $\overline{U} = X$) and U is irreducible.

(c) If X is irreducible and $f \colon X \to Y$ is continuous then $f(X)$ is irreducible.

(d) If Y is irreducible, $Y \subset X$, and $\overline{Y}$ is the closure of Y in X, then $\overline{Y}$ is irreducible.

(e) If X has an open covering $X = \bigcup U_i$ where each U_i is irreducible and $U_i \cap U_j \neq \emptyset$ for all i and j, then X is irreducible.

(2) Let $f \colon X \to Y$ be a morphism of affine varieties and

$$f^* \colon k[Y] \to k[X], \quad g \mapsto g \circ f$$

the corresponding morphism of k-algebras. Recall that, for $J \subset k[X]$ an ideal we have $I(V(J)) = \sqrt{J}$, the radical of J (Hilbert's Nullstellensatz). Verify the following statements.

(a) For $Z \subset X$ a subset, $V(I(Z)) = \overline{Z}$ (the closure of Z in the Zariski topology).

(b) For $J \subset k[Y]$ an ideal, $f^{-1}V(J) = V(f^*(J))$.
[Note: $f^*(J)$ is not necessarily an ideal of $k[X]$. But we can define $V(S)$ for any subset S of $k[X]$, then $V(S) = V(\langle S \rangle)$ where $\langle S \rangle \subset k[X]$ is the ideal generated by S.]

(c) For $Z \subset X$ a subset we have $I(f(Z)) = f^{*-1}I(Z)$. In particular (using the case Z is a point), the map of sets $f\colon X \to Y$ corresponds to the map $\mathfrak{m} \mapsto f^{*-1}(\mathfrak{m})$ from maximal ideals of $k[X]$ to maximal ideals of $k[Y]$.

(d) For $J \subset k[X]$ an ideal, $\overline{f(V(J))} = V(f^{*-1}J)$. In particular, $\overline{f(X)} = Y$ iff f^* is injective.

(3) Let X be the union of the coordinate axes in $\mathbb{A}^3$.

(a) Compute the ideal $I(X) \subset k[x,y,z]$.

(b) Prove that $I(X)$ cannot be generated by 2 elements.

(c) Let $J = (xy, (x-y)z) \subset k[x,y,z]$. Show that $V(J) = X$. What is $\sqrt{J}$?

(4) Let $J = (x^2+y^2+z^2, xy+yz+xz) \subset k[x,y,z]$ and $X = V(J) \subset \mathbb{A}^3$. Determine the irreducible components of X. What is $\sqrt{J}$?

(5) Let X be an affine variety and $f \in k(X)$ a rational function on X. Define

$$\operatorname{domain}(f) = \{p \in X \mid f \in \mathcal{O}_{X,p}\}.$$

(a) Prove $\operatorname{domain}(f) \subset X$ is an open subset.

(b) Let $p \in X$. Suppose $f = g/h$, where $g, h \in k[X]$, and $g(p) \neq 0$, $h(p) = 0$. Show that $p \notin \operatorname{domain}(f)$.

(c) Compute $\operatorname{domain}(f)$ in the following cases:

i. $X = V(x_1^2 + x_2^2 - 1) \subset \mathbb{A}^2$, $f = (1-x_2)/x_1$.

ii. $X = V(x_1x_3 - x_2^2) \subset \mathbb{A}^3$, $f = x_1/x_2$.

(6) Consider the morphism

$$f\colon \mathbb{A}^1 \to \mathbb{A}^2, \quad t \mapsto (t^2, t^3).$$

(a) Show that $X := f(\mathbb{A}^1) \subset \mathbb{A}^2$ is closed and find its ideal $I(X) \subset k[x,y]$.

(b) Draw a sketch of X in the case $k = \mathbb{R}$, and observe that the origin is a singular point of X (there is no well-defined tangent line).

[WARNING: In general we don't allow non-algebraically closed fields, but it is sometimes useful for visualization to consider $k = \mathbb{R}$.]

(c) One can also try to draw a (partial) sketch in the case $k = \mathbb{C}$ as follows. Consider the intersection of X with a small sphere $S^3 \subset \mathbb{C}^2$ with center the origin. Show that the intersection $X \cap S^3$ is a trefoil knot in $S^3 = \mathbb{R}^3 \cup \{\infty\}$. This shows in particular that the origin is a singular point of X (otherwise $X \cap S^3 \subset S^3$ would be an unknotted S^1).

[Hint: The intersection $X \cap S^3$ lies on one of the tori $S^1 \times S^1$ in S^3 defined by $|x| = a$, $|y| = \sqrt{r^2 - a^2}$ for some $0 < a < r$, where r is the radius of the sphere.]

(d) Show that the map $\mathbb{A}^1 \to X$ is a homeomorphism of topological spaces (for the Zariski topology).

(e) Show that, via f^*, the coordinate ring $k[X]$ is identified with the subring of the polynomial ring $k[\mathbb{A}^1] = k[t]$ consisting of polynomials $g(t)$ such that $g'(0) = 0$.

(f) Show that f^* defines an isomorphism of the function fields $k(X) \xrightarrow{\sim} k(\mathbb{A}^1) = k(t)$.

(g) Using (e) and (f) or otherwise, determine the integral closure of $k[X]$.

(7) Assume $\mathrm{char}(k) \neq 2$. Consider the morphism

$$f \colon \mathbb{A}^2 \to \mathbb{A}^3, \quad (x_1, x_2) \mapsto (x_1^2, x_1 x_2, x_2^2).$$

(a) Prove that $X := f(\mathbb{A}^2) \subset \mathbb{A}^3$ is closed and find its ideal $I(X) \subset k[y_1, y_2, y_3]$.

(b) Show that, as a topological space (for the Zariski topology), X is the quotient of $\mathbb{A}^2$ by the action of $\mathbb{Z}/2\mathbb{Z}$ given by $(x_1, x_2) \mapsto (-x_1, -x_2)$.

(c) Show that, via f^*, the coordinate ring $k[X]$ is identified with the invariant ring $k[x_1, x_2]^{\mathbb{Z}/2\mathbb{Z}}$ of the group action on the coordinate ring $k[x_1, x_2]$ of $\mathbb{A}^2$. [Here for a group G acting on a ring R the *invariant ring* R^G is the subring of R consisting of elements r such that $g \cdot r = r$ for all $g \in G$.] This implies that X is the quotient of $\mathbb{A}^2$ by the group action as an algebraic variety.

[Hint / Remark: If $f \colon X \to Y$ is a morphism of affine varieties, we say that f is a *finite morphism* if the corresponding homomorphism of k-algebras $f^* \colon k[Y] \to k[X]$ gives $k[X]$ the structure of a finitely generated $k[Y]$-module. In this case, it follows from the "going up theorem" (cf. 612) that the morphism f is *closed*, that is, if $Z \subset X$ is closed then $f(Z) \subset Y$ is closed. Moreover, if $f \colon X \to Y$ is a finite morphism then $f^{-1}(p)$ is a finite set for all $p \in Y$. The morphisms f in questions 6 and 7 above are examples of finite morphisms.]

(8) For each of the following morphisms $f \colon X \to Y$, compute the image $f(X) \subset Y$ of f. Show that $f(X)$ is neither open nor closed in Y, and $\overline{f(X)} = Y$. Describe the fiber $f^{-1}(p)$ of f over each point $p \in Y$.

   (a) $f \colon \mathbb{A}^2 \to \mathbb{A}^2$, $(x, y) \mapsto (x, xy)$.

   (b) $f \colon \mathbb{A}^3 \to \mathbb{A}^3$, $(x, y, z) \mapsto (x, xy, xyz)$.

(9) (a) Let $J \subset S = k[X_0, \ldots, X_n]$ be a homogeneous ideal. Show that if J is not prime then there exist *homogeneous* elements $a, b \in S$ such that $ab \in J$ and $a, b \notin J$.

   (b) Let $X \subset \mathbb{P}^n$ be an algebraic set. Show that X is irreducible iff $I(X) \subset S$ is prime.

(10) Let

$$X = V(x_1^3 + x_1 x_2^2 + x_1^2 + x_2 + 1) \subset \mathbb{A}^2.$$

Let $\overline{X}$ denote the closure of X in

$$\mathbb{P}^2 = (X_0 \neq 0) \sqcup (X_0 = 0) = \mathbb{A}^2 \sqcup \mathbb{P}^1.$$

   (a) Write down the homogeneous equation of $\overline{X}$ and identify the set $\overline{X} \setminus X = \overline{X} \cap \mathbb{P}^1$.

   (b) Find another affine chart $Y \subset \mathbb{A}^2$ for $\overline{X}$ such that $\overline{X} = X \cup Y$, write down the equation of $Y \subset \mathbb{A}^2$, and describe the transition map between the two charts explicitly.

(11) Let $F \in k[X_0, X_1, X_2]$ be an irreducible homogeneous polynomial of degree d. Let $X = V(F) \subset \mathbb{P}^2$ be the associated projective variety, a projective plane curve of degree d. Let $L \subset \mathbb{P}^2$ be a line (that is, $L = V(a_0X_0 + a_1X_1 + a_2X_2) \subset \mathbb{P}^2$ is the zero locus of a linear form). Show that $X \cap L$ consists of exactly d points counting multiplicities (unless $d = 1$ and $X = L$).

(12) Show directly using the standard affine charts that $\mathcal{O}_X(X) = k$ for $X = \mathbb{P}^1$.

(13) Let $X = V(f) \subset \mathbb{A}^2$. Suppose

$$f = a_1x_1 + a_2x_2 + \cdots$$

where $\cdots$ denotes higher order terms in x_1, x_2, and $(a_1, a_2) \neq (0, 0)$. (Geometrically, we have $(0, 0) \in X$, and X is smooth at $(0, 0)$ with tangent line $V(a_1x_1 + a_2x_2) \subset \mathbb{A}^2$.) Consider the morphism

$$q\colon \mathbb{A}^2 \setminus \{(0,0)\} \to \mathbb{P}^1, \quad (x_1, x_2) \mapsto (x_1 : x_2).$$

   (a) Show that the restriction of q to $X \setminus \{(0, 0)\}$ extends to a morphism $g\colon X \to \mathbb{P}^1$.
   (b) What is the geometric interpretation of the point $g(0, 0) \in \mathbb{P}^1$?

(14) Let $n \in \mathbb{Z}$. Let $X = X(n) = U_1 \cup U_2$ where $U_1 = \mathbb{A}^2_{x_1,y_1}$, $U_2 = \mathbb{A}^2_{x_2,y_2}$, and the glueing is given by

$$U_1 \supset (x_1 \neq 0) \xrightarrow{\sim} (x_2 \neq 0) \subset U_2, \quad (x_1, y_1) \mapsto (x_1^{-1}, x_1^n y_1).$$

   (a) Show that $C \subset X$ defined by $C \cap U_i = V(y_i)$ for $i = 1, 2$ is a closed subvariety isomorphic to $\mathbb{P}^1$.
   (b) Show that the morphisms

$$p_i\colon U_i \to \mathbb{A}^1, \quad (x_i, y_i) \mapsto x_i$$

   patch to give a morphism $p\colon X \to \mathbb{P}^1$. Moreover there is a morphism $s\colon \mathbb{P}^1 \to X$ such that $p \circ s = \mathrm{id}_{\mathbb{P}^1}$ and $s(\mathbb{P}^1) = C$.
   (c) Compute $\mathcal{O}_X(X)$ as a subring of $k[x_1, y_1]$. For $n < 0$, show that $\mathcal{O}_X(X) = k$. For $n \geq 0$, find an explicit set of $n + 1$ generators for $\mathcal{O}_X(X)$ as a k-algebra.

(d) Let $f\colon X \to \mathbb{A}^{n+1}$ be the morphism defined by the generators for $\mathcal{O}_X(X)$ found in (c). Show that $f(X)$ is closed, $f(C)$ is a point, and the restriction of f to $X \setminus C$ is an isomorphism.

[Hint: If you are stuck, try $n = 1$ and $n = 2$ first.]