Math 797AS Homework 2

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(1) Recall the *maximum principle* from MATH 621: Suppose $\Omega \subset \mathbb{C}$ is a connected open set and $f\colon \Omega \to \mathbb{C}$ is a holomorphic function. If $|f|$ has a maximum in Ω then f is constant. Use the maximum principle to prove the following statement: if X is a connected compact complex manifold and $f\colon X \to \mathbb{C}$ is a global holomorphic function then f is constant.

(2) Recall that if X and Y are topological spaces, $f\colon X \to Y$ is a continuous map, and $\mathcal{F}$ is a sheaf on X, then the *pushforward* $f_*\mathcal{F}$ is the sheaf on Y defined by $f_*\mathcal{F}(U) = \mathcal{F}(f^{-1}U)$.

   (a) Let X be a topological space, $p \in X$ a closed point, and A an abelian group. Let $\mathcal{F}$ be the sheaf on X defined by

   $$\mathcal{F}(U) = \begin{cases} A & \text{if } p \in U \\ 0 & \text{otherwise.} \end{cases}$$

   ($\mathcal{F}$ is called the *skyscraper sheaf* supported at p with stalk A.) Show that $H^k(X, \mathcal{F}) = 0$ for $k > 0$.

   (b) Let X be a topological space and $Y \subset X$ a closed subset (with the induced topology). Write $i\colon Y \to X$ for the inclusion. Let $\mathcal{G}$ be a sheaf on Y. Show that $H^k(X, i_*\mathcal{G}) = H^k(Y, \mathcal{G})$ for all k.

   [Remark: In part (b), for the special case that Y is a point the sheaf $\mathcal{F} = i_*\mathcal{G}$ on X is a skyscraper sheaf supported at Y as in part (a).]

(3) Let X be a compact complex curve (a Riemann surface). Let $D = \sum_{i=1}^r n_i p_i$ be a finite formal sum of points of X with multiplicities

$n_i \in \mathbb{N}$. Define the sheaf $\mathcal{O}_X(D)$ on X as follows: Let $\Gamma(U, \mathcal{O}_X(D))$ be the set of meromorphic functions f on U such that f has a pole of order $\leq n_i$ at p_i for each $p_i \in U$ and is holomorphic elsewhere.

(a) Show that $\mathcal{O}_X(D)$ is a locally free sheaf of rank 1. (So $\mathcal{O}_X(D)$ is the sheaf of holomorphic sections of a holomorphic line bundle $p: L \to X$.)

(b) Let z_i be a local coordinate at $p_i \in X$. Show that there is an exact sequence of sheaves on X

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(D) \to \bigoplus_{i=1}^{r} \mathbb{C}[z_i]/(z_i^{n_i}) \to 0$$

where the last term is a direct sum of skyscraper sheaves at the points p_i with stalks $\mathbb{C}[z_i]/(z_i^{n_i})$.

(4) Recall that a closed complex submanifold Y of dimension $n - k$ of a complex manifold X of dimension n is a closed subset $Y \subset X$ such that there exist charts $\varphi_i: U_i \to \mathbb{C}^n_{z_1, \dots, z_n}$ for X satisfying $Y \subset \bigcup U_i$ and

$$\varphi_i(Y \cap U_i) = \varphi_i(U_i) \cap \{(z_1, \dots, z_n) \mid z_1 = \dots = z_k = 0\}$$

for each i. (Then the restrictions of the charts φ_i give charts for Y as a complex manifold of dimension $n - k$.)

(a) Let Y be a closed complex submanifold of a complex manifold X. The *ideal sheaf* of $Y \subset X$ is the sheaf of holomorphic functions on X which vanish along Y, that is,

$$\mathcal{I}_Y(U) = \{f \in \mathcal{O}_X(U) \mid f|_{Y \cap U} = 0\}.$$

Let $i: Y \to X$ be the inclusion. Show that there is an exact sequence of sheaves on X

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to i_*\mathcal{O}_Y \to 0.$$

(b) Show from first principles that $i_*\mathcal{O}_Y$ and $\mathcal{I}_Y$ are coherent.

[Remark: More generally, one has the following theorem of Grauert: Suppose X and Y are complex manifolds, $\mathcal{F}$ is a coherent sheaf on X, and $f: X \to Y$ is a holomorphic map. If f is proper (that is, the inverse image of a compact set is compact) then $f_*\mathcal{F}$ is a coherent sheaf on Y.]

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(5) Let $X = \mathbb{C}$ and $Z = \{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\} \subset X$, a closed subset. Show that the ideal sheaf $\mathcal{I}_Z \subset \mathcal{O}_X$ defined by

$$\mathcal{I}_Z(U) = \{f \in \mathcal{O}_X(U) \mid f|_{Z \cap U} = 0\}$$

is *not* coherent.

(6) For X a complex manifold, $\mathcal{F}$ a coherent sheaf on X, and $p \in X$ a point, define the *fiber* of $\mathcal{F}$ at p to be the finite dimensional $\mathbb{C}$-vector space

$$\mathcal{F} \otimes \mathbb{C}(p) := \mathcal{F}_p \otimes_{\mathcal{O}_{X,p}} \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}.$$

Here $\mathfrak{m}_{X,p} = \{f \in \mathcal{O}_{X,p} \mid f(p) = 0\}$ is the maximal ideal of the local ring $\mathcal{O}_{X,p}$; evaluation at p defines an isomorphism $\mathcal{O}_{X,p}/\mathfrak{m}_{X,p} \to \mathbb{C}$.

(a) Let $p\colon E \to X$ be a holomorphic vector bundle over a complex manifold X, $\mathcal{E}$ the sheaf of holomorphic sections of E, and $q \in X$ a point. Show that the fiber of $\mathcal{E}$ at q is identified with the fiber $E_q := p^{-1}(q)$ of $p\colon E \to X$ at q.

(b) Let $\mathcal{F}$ be a coherent sheaf on a complex manifold X. Show that for $m \in \mathbb{Z}_{\geq 0}$ the subset

$$Z_m = \{p \in X \mid \dim_{\mathbb{C}}(\mathcal{F} \otimes \mathbb{C}(p)) \geq m\}$$

is a closed analytic subset (that is, locally defined by the vanishing of a finite set of holomorphic functions). In particular, the function

$$X \to \mathbb{Z}, \quad p \mapsto \dim_{\mathbb{C}}(\mathcal{F} \otimes \mathbb{C}(p)) \qquad (*)$$

is *upper semi-continuous*.

[Hints: Recall the definition of a coherent sheaf. Tensor product is right exact. The rank of a matrix A is $< k$ iff the $k \times k$ minors vanish.]

(c) With notation as in part (b), suppose X is connected. Show that $\mathcal{F}$ is locally free iff the function $(*)$ is constant.

[Hint: The ring $\mathcal{O}_{X,p}$ is Noetherian. Use Nakayama's lemma.]

(7) Let $\mathbb{P}^n$ be complex projective space of dimension n. For $d \in \mathbb{Z}$, let $p\colon L_d \to \mathbb{P}^n$ the holomorphic line bundle defined as follows: Writing

$X_0, \ldots, X_n$ for the homogeneous coordinates on $\mathbb{P}^n$, we have the open sets $U_i = (X_i \neq 0) \simeq \mathbb{C}^n$. Then L_d has local trivializations

$$\varphi_i \colon p^{-1}(U_i) \xrightarrow{\sim} U_i \times \mathbb{C}$$

with transition functions $g_{ij} \colon U_i \cap U_j \to \mathbb{C}^\times$ given by $g_{ij} = (X_i/X_j)^d$. Here recall that that *transition function* g_{ij} is defined by

$$\varphi_j \circ \varphi_i^{-1} \colon U_i \cap U_j \times \mathbb{C} \xrightarrow{\sim} U_i \cap U_j \times \mathbb{C}, \quad (p, v) \mapsto (p, g_{ij}(p) \cdot v).$$

Let $\mathcal{O}_{\mathbb{P}^n}(d)$ denote the sheaf of holomorphic sections of $L_d \to \mathbb{P}^n$. Using the open covering $\mathbb{P}^n = \bigcup U_i$ or otherwise, show that the global sections $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ can be identified with the complex vector space of homogeneous polynomials of degree d in $X_0, \ldots, X_n$.

[Remark: The line bundles L_d can be described intrinsically as follows: Let $L \subset \mathbb{P}^n \times \mathbb{C}^{n+1}$ be the *tautological line bundle* over $\mathbb{P}^n$ whose fiber over a point $[v] \in \mathbb{P}^n = \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^\times$ is the associated line $l = \mathbb{C} \cdot v \subset \mathbb{C}^{n+1}$. Then $L_d = (L^*)^{\otimes d}$ for $d > 0$, and $L_d = L^{\otimes(-d)}$ for $d < 0$ (and L_0 is trivial).]

(8) Let $\mathbb{P}^n$ be complex projective n-space with homogeneous coordinates $X_0, \ldots, X_n$. Let F be a homogeneous polynomial of degree d in $X_0, \ldots, X_n$ and
$$X = \{p \in \mathbb{P}^n \mid F(p) = 0\} \subset \mathbb{P}^n$$
the hypersurface defined by F. Assume that dF is nowhere zero on $\mathbb{C}^{n+1} \setminus \{0\}$, so that X is a complex manifold of dimension $n-1$ by the implicit function theorem.

(a) Show that the ideal sheaf $\mathcal{I}_X \subset \mathcal{O}_{\mathbb{P}^n}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(-d)$.

(b) One has the following result on the cohomology of the sheaves $\mathcal{O}_{\mathbb{P}^n}(m)$:

- $H^0(\mathcal{O}_{\mathbb{P}^n}(m))$ is identified with the space of homogeneous polynomials of degree m in $X_0, \ldots, X_n$,
- $H^k(\mathcal{O}_{\mathbb{P}^n}(m)) = 0$ for $0 < k < n$, and
- $H^n(\mathcal{O}_{\mathbb{P}^n}(m)) \simeq H^0(\mathcal{O}_{\mathbb{P}^n}(-m-n-1))^*$.

This can be proved using Cech cohomology for the Stein open covering of $\mathbb{P}^n$ given by the $U_i = (X_i \neq 0) \simeq \mathbb{C}^n$. See e.g. [H77],

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Chapter III, Theorem 5.1, p. 225 for a proof in the algebraic category (which implies the result in the analytic category by [S55]). Now, with notation as above, deduce that $H^k(\mathcal{O}_X) = 0$ for $0 < k < n - 1$.

(9) Let F and G be irreducible homogeneous polynomials in X_0, X_1, X_2 of degrees d and e. Let $X = (F = 0) \subset \mathbb{P}^2$ and $Y = (G = 0) \subset \mathbb{P}^2$ be the associated irreducible complex curves (possibly singular) in the complex projective plane. Assume $X \neq Y$, then $X \cap Y$ is finite. For $p \in X \cap Y$, define the *intersection multiplicity* of X and Y at p by

$$(X \cdot Y)_p = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^2, p}/(f_p, g_p)$$

where $f_p, g_p \in \mathcal{O}_{\mathbb{P}^2, p}$ are local equations for X and Y.

(a) Show that there is an exact sequence of sheaves on $\mathbb{P}^2$

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d-e) \to \mathcal{O}_{\mathbb{P}^2}(-d) \oplus \mathcal{O}_{\mathbb{P}^2}(-e) \to \mathcal{O}_{\mathbb{P}^2} \to \bigoplus_{p \in X \cap Y} \mathcal{O}_{\mathbb{P}^2, p}/(f_p, g_p) \to 0$$

where the last term is a direct sum of skyscraper sheaves supported at the intersection points of X and Y.

[Hints: (1) If Z is a complex manifold, $p \in Z$ is a point, and $z_1, \ldots, z_n$ are local coordinates at p, then $\mathcal{O}_{Z,p}$ is the ring $\mathbb{C}\{z_1, \ldots, z_n\}$ of convergent complex power series in $z_1, \ldots, z_n$. This ring is a UFD, see e.g. Griffiths and Harris, p. 10.

(2) We have $\mathcal{I}_X \simeq \mathcal{O}_{\mathbb{P}^2}(-d)$, cf. Q7(a). Here, if $p \in X$ is a singular point, one needs to use e.g. Griffiths and Harris, p. 11–12 to show that if $p \in (X_i \neq 0)$ then $f_p := F/X_i^d$ generates $\mathcal{I}_{X,p}$.)]

(b) Deduce *Bézout's theorem*:

$$X \cdot Y := \sum_{p \in X \cap Y} (X \cdot Y)_p = d \cdot e.$$

[Hints: (1) In any abelian category, a long exact sequence

$$0 \to A_m \xrightarrow{\theta_m} \cdots \xrightarrow{\theta_2} A_1 \xrightarrow{\theta_1} A_0 \to 0$$

can be divided into short exact sequences

$$0 \to \ker \theta_i \to A_i \to \operatorname{im} \theta_i \to 0$$

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(note $\operatorname{im} \theta_i = \ker \theta_{i-1}$ by assumption). In particular, it follows that, for a long exact sequence

$$0 \to \mathcal{F}_m \xrightarrow{\theta_m} \cdots \xrightarrow{\theta_2} \mathcal{F}_1 \xrightarrow{\theta_1} \mathcal{F}_0 \to 0$$

in the category of coherent sheaves on a compact complex manifold X, we have $\sum_{k=0}^m (-1)^k \chi(X, \mathcal{F}_k) = 0$ (why?).
(2) $\chi(\mathcal{O}_{\mathbb{P}^2}(m))$ can be computed using the statement in Q7(b).]

References

[H77]   R. Hartshorne, Algebraic geometry, Grad. Texts in Math. 52. Springer-Verlag, 1977.

[S55]   J-P. Serre, Géométrie algébrique et géométrie analytique, Ann. Inst. Fourier, Grenoble 6 (1955–1956), 1–42.