1. a) Recall the Cauchy integral formula for derivatives :-

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw$$

$$\Rightarrow \quad |f^{(n)}(z_0)| \le \frac{n!}{2\pi}\cdot \text{length}(\gamma)\cdot \sup_{w\in\gamma}\left|\frac{f(w)}{(w-z_0)^{n+1}}\right|$$

$$= \frac{n!}{2\pi}\cdot 2\pi R\cdot \frac{M}{R^{n+1}},$$

i.e. $|f^{(n)}(z_0)| \le \frac{n!\cdot M}{R^n}$. — Cauchy inequality.

b) Let $f(z) = \frac{M}{R^n}\cdot(z-z_0)^n$

Then $|f(z)| = M$ for $z\in\gamma$

& $|f^{(n)}(z_0)| = \left|\frac{n!\,M}{R^n}\right| = \frac{n!\,M}{R^n}$

2. Recall $e^z = \sum_{k=0}^{\infty}\frac{z^k}{k!}$ (valid for all $z\in\mathbb{C}$)

& $e^z \neq 0 \quad \forall\, z\in\mathbb{C}$.

$\therefore\ P_n(z) = \sum_{k=0}^{n}\frac{z^k}{k!} \longrightarrow e^z$ as $n\to\infty$,

& the convergence is uniform on the disc $\{|z|\le R\}$ for any fixed R.

(More generally, if $\sum_{k=0}^{\infty} a_k z^k$ is a power series w/ radius of convergence R, then $\sum_{k=0}^{n} a_k z^k \longrightarrow \sum_{k=0}^{\infty} a_k z^k$ as $n\to\infty$ for $|z|<R$, and the convergence is uniform on the disc $\{|z|\le R'\}$

for any fixed $R' < R$).

Given R, we have $|e^z| \geq \epsilon > 0$ for $|z| \leq R$, some $\epsilon > 0$. (because a continuous function on a compact set is bounded & attains its bounds).

Given $\epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ $|P_n(z) - e^z| < \epsilon$ & $\forall |z| \leq R$

(uniform convergence on $|z| \leq R$).

Then $|P_n(z)| \geq \big|\, |P_n(z) - e^z| - |e^z| \,\big| > 0$ for $n \geq N$ & $|z| \leq R$,

i.e. $P_n(z) \neq 0$ for $n \geq N$ & $|z| \leq R$.

3. Let $g(z) = e^{f(z)}$

Then $g : \mathbb{C} \to \mathbb{C}$ hol

& $|g| = \cancel{e^{|Re(f)|}}\ e^{Re(f)} \leq e^M$

where $Re(f) \leq M$. [recall $e^z = e^{x+iy} = e^x \cdot e^{iy}$
$= e^x \cdot (\cos y + i \sin y)$
$\Rightarrow |e^z| = e^x$]

By Liouville's theorem, g is constant.

Recall: $e^{z_1} = e^{z_2} \iff z_1 - z_2 = (2\pi i) \cdot k$, some $k \in \mathbb{Z}$.

Thus $f : \mathbb{C} \to \mathbb{C}$ takes values in a discrete set.
Since f is continuous (& its domain $\mathbb{C}$ is connected)
we find that f is constant.

4 a. $\dfrac{\sin z}{z(z-\pi/2)^2}$ has singularities at $z=0, \pi/2$
(the zeroes of the denominator)

$z=0$ is removable : $\dfrac{\sin z}{z(z-\pi/2)^2} = \dfrac{z - z^3/3! + z^5/5! - \cdots}{z(z-\pi/2)^2}$

$= \dfrac{1 - z^2/3! + z^4/5! - \cdots}{(z-\pi/2)^2}$, hol. at $z=0$.

$z=\pi/2$ is a pole of order 2 : $\dfrac{\sin z}{z(z-\pi/2)^2} = \dfrac{(\sin z/z)}{(z-\pi/2)^2}$ ← hol & $\neq 0$ at $z=\pi/2$.

$\operatorname{res}_{z=0} \dfrac{\sin z}{z(z-\pi/2)^2} = 0$ (removable.)

$\operatorname{res}_{z=\pi/2} \dfrac{\sin z}{z(z-\pi/2)^2} = \lim\limits_{z\to\pi/2} \dfrac{d}{dz}\left(\dfrac{\sin z}{z}\right)$

$= \lim\limits_{z\to\pi/2} \left(\dfrac{\cos z \cdot z - \sin z \cdot 1}{z^2}\right) = -1/(\pi/2)^2.$

b. $z^2 e^{1/(z+1)}$

has singularity at $z=-1$.

This is an essential singularity:-

$z^2 e^{1/(z+1)} = ((z+1)-1)^2 e^{1/z+1}$

$= (+1 - 2(z+1) + (z+1)^2) \cdot \sum\limits_{k=0}^{\infty} \dfrac{1}{k!}(z+1)^{-k}$

$= \sum\limits_{k=1}^{\infty} \left(\dfrac{1}{k!} \dfrac{-2}{(k+1)!} + \dfrac{1}{(k+2)!}\right) \cdot (z+1)^{-k} + $ (holomorphic)

$\neq 0$ for $k \geq 1$. $\Rightarrow$ essential.

$\operatorname{res}_{z=-1} z^2 e^{1/z+1}$ = coefficient of $(z+1)^{-1}$ in Laurent series expansion

$= \left(\dfrac{1}{1!} \dfrac{-2}{2!} + \dfrac{1}{3!}\right) = 1/6$

(Note: In general, if f has an essential singularity at z_0 & g has a pole or is holomorphic at z_0, (& g is not identically zero) then $f \cdot g$ has an essential singularity at z_0).

c. $(\cot z)^2 = \left(\frac{\cos z}{\sin z}\right)^2$

Recall (HW2 Q5) that $\sin z = 0 \iff z = k\cdot\pi,\ k \in \mathbb{Z}$, & these are zeroes of order 1 (i.e. $(\sin z)'\big|_{z=k\pi} \neq 0$)

Also $\cos z\big|_{z=k\pi} = (-1)^k \neq 0$

So $(\cot z)^2$ has poles of order 2 at $z = k\pi,\ k \in \mathbb{Z}$ (& no other singularities.

To compute the residues:-

Observe that, by the addition formulae for $\sin z$ & $\cos z$ (HW1 Q6), ~~(or directly from $e^{iz} = \cos z + i \sin z$~~

$\sin(z+\pi) = -\sin z, \quad \cos(z+\pi) = -\cos z,$

so $\cot(z+\pi) = \cot(z)$

Thus all the residues are equal, & WMA $z=0$.

$(\cot z)^2 = a_{-2} z^{-2} + a_{-1} z^{-1} + \cdots, \quad \operatorname{res}_{z=0} (\cot z)^2 = a_{-1}$

In fact, $(\cot z)^2$ is even, i.e., $(\cot(-z))^2 = (\cot z)^2$

so we must have $a_k = 0$ for k odd

& in particular $\operatorname{res}_{z=0} (\cot z)^2 = 0$.

d. $\frac{z^{35}}{1-z^{16}}$ has simple poles (order 1) at $z = e^{2\pi i k/16}$, $k = 0,1,2,\ldots,15$

(the zeroes of the denominator).

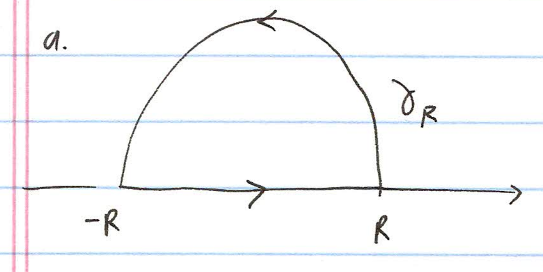
$$\operatorname{res}_{z=e^{2\pi i k/16}} \overset{\dagger}{=} \frac{(z^{35})\big|_{z=e^{2\pi i k/16}}}{(1-z^{16})'\big|_{z=e^{2\pi i k/16}}} = \frac{e^{2\pi i k \cdot 35/16}}{-16 \cdot e^{2\pi i k \cdot 15/16}} =$$

† if $f(z) = g(z)/h(z)$, g,h hol at z_0, $g(z_0) \neq 0$, $h(z_0) = 0$, $h'(z_0) \neq 0$ then $\operatorname{res}_{z_0} f(z) = \frac{g(z_0)}{h'(z_0)}$

$$= \frac{-1}{16}\, e^{2\pi i k \cdot 20/16} = -1/16\, e^{2\pi i k \cdot 5/4} = -1/16 \cdot e^{2\pi i k \cdot 1/4}$$

$$= -1/16 \cdot \left(e^{2\pi i/4}\right)^k = \frac{-1}{16} \cdot i^k.$$

5. a.



γ_R semicircle with diameter $[-R, R] \subset \mathbb{R}$, oriented ccw.

$$\underbrace{\int_{\gamma_R} \frac{1}{z^6+1}\,dz}_{\substack{\| \text{ R.T.} \\ 2\pi i \cdot \sum \text{ residues of } 1/z^6+1 \\ \text{at poles inside } \gamma_R}} = \int_{-R}^{R} \frac{1}{x^6+1}\,dx + \underbrace{\int_{\frown} \frac{1}{z^6+1}\,dz}_{\xrightarrow{(*)}\ 0 \text{ as } R\to\infty}$$

(*): $$\left| \int_{\frown} \frac{1}{z^6+1}\,dz \right| \le \underbrace{(\pi R)}_{\text{length}(\frown)} \cdot \underbrace{\frac{1}{R^6-1}}_{\text{upper bound for } \left|\frac{1}{z^6+1}\right| \text{ on } |z|=R} \longrightarrow 0 \text{ as } R \to \infty.$$

$\frac{1}{z^6+1}$ has simple poles at $z = e^{2\pi i/6 \cdot k + \pi i/6}$ (zeros of z^6+1)

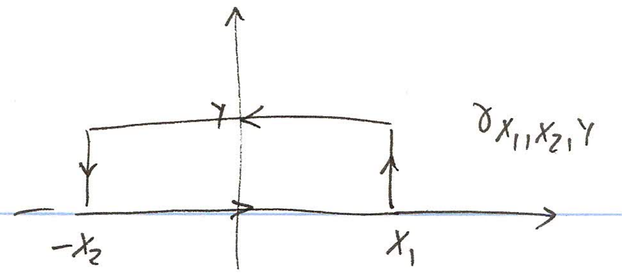
There are 3 poles inside γ_R $(R>1)$: $e^{\pi i/6} = \cancel{\frac{1}{2} + \frac{\sqrt{3}}{2} i} = \sqrt{3}/2 + 1/2\, i$, $e^{\pi i/2} = i$, $e^{5\pi i/6} = \cancel{-\frac{1}{2} + \frac{\sqrt{3}}{2} i} = -\sqrt{3}/2 + \frac{1}{2} i$

So $$\int_{-\infty}^{\infty} \frac{1}{x^6+1}\,dx = \lim_{R\to\infty} \int_{\gamma_R} \frac{1}{z^6+1}\,dz$$

$$= 2\pi i \cdot \sum_{\substack{z_0 = e^{\pi i/6}, e^{\pi i/2}, \\ e^{5\pi i/6}}} \operatorname{res}_{z_0} \frac{1}{z^6+1}$$

$$= 2\pi i \cdot \sum_{z_0} \frac{1}{6 z_0^5} \overset{z_0^6 = -1}{=} \frac{2\pi i}{6} \cdot \sum_{z_0} -z_0 = \frac{-2\pi i}{6}(2i) = \boxed{\frac{2\pi}{3}}.$$

b.



$$\frac{z^3\sin z}{z^4+5z^2+4} = \mathrm{Im}\left(\frac{z^3\cdot e^{iz}}{z^4+5z^2+4}\right) \quad \text{for } z=x\in\mathbb{R}.$$

$$:- \quad e^{iz} = \cos z + i\sin z.$$

$$2\pi i\cdot\left(\sum \begin{array}{c}\text{residues}\\ \text{inside } \gamma_{X_1,X_2,Y}\end{array}\right) \overset{R.T.}{=} \int_{\gamma_{X_1,X_2,Y}} \frac{z^3\cdot e^{iz}}{z^4+5z^2+4} = \int_{-X_2}^{X_1} \frac{x^3\cdot e^{ix}}{x^4+5x^2+4}\,dx$$

$$+ \int_{\uparrow} \frac{z^3\cdot e^{iz}}{z^4+5z^2+4}\,dz + \int_{\leftarrow} \text{''} + \int_{\downarrow} \text{''}.$$

$$\left|\int_{\uparrow}\right| \le \frac{C}{X_1}\cdot\int_0^Y e^{-y}\,dy < \frac{C}{X_1}$$

Similarly $\left|\int_{\downarrow}\right| < \frac{C}{X_2}$

$$\left|\int_{\leftarrow}\right| \le (X_1+X_2)\cdot\frac{C}{Y}\cdot e^{-Y}$$

Here, we've used $\left|\frac{z^3}{z^4+5z^2+4}\right| \le C\cdot\frac{|z|^3}{|z|^4} \le \frac{C}{R}$ for $|z|\geq R$

for some constant C, & R sufficiently large

$$\& \quad |e^{iz}| = e^{\mathrm{Re}(iz)} = e^{-y}.$$

Now let $Y\to\infty$ (with X_1,X_2 fixed), & then $X_1,X_2\to\infty$, to obtain

$$\cancel{\oint_{\gamma_{X_1,X_2}}} \quad \cancel{\mathrm{Im}}\ 2\pi i\cdot\sum\left(\begin{array}{c}\text{residues in}\\ \text{upper half plane}\end{array}\right) = \int_{-\infty}^{\infty}\frac{x^3 e^{ix}}{x^4+5x^2+4}\,dx$$

$$z^4+5z^2+4 = (z^2+1)(z^2+4)$$

So $\frac{z^3\cdot e^{iz}}{z^4+5z^2+4}$ has simple poles at $z=\pm i, \pm 2i$

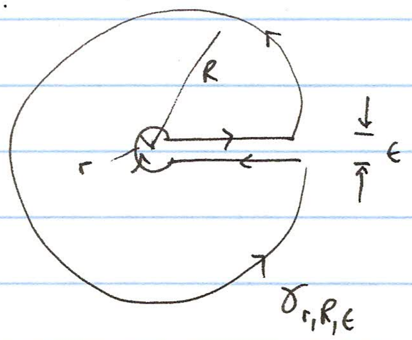
$$\mathrm{res}_{z=i}\left(\frac{z^3\cdot e^{iz}}{z^4+5z^2+4}\right) = \frac{z^3\cdot e^{iz}\big|_{z=i}}{4z^3+10z\big|_{z=i}} = \frac{-i\cdot e^{-1}}{-4i+10i} = \frac{-e^{-1}}{6}.$$

$$\operatorname{res}_{z=2i} = \frac{z^3 \cdot e^{iz}\big|_{z=2i}}{4z^3+10z\big|_{z=2i}} = \frac{-8i\cdot e^{-2}}{-32i+20i} = \frac{2}{3}\cdot e^{-2}$$

Thus
$$\int_{-\infty}^{\infty} \frac{x^3 e^{ix}}{x^4+5x^2+4}\,dx = 2\pi i\left(-\frac{e^{-1}}{6}+\frac{2}{3}e^{-2}\right)$$

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{x^4+5x^2+4}\,dx = \operatorname{Im}(\cdots) = 2\pi\cdot\left(-\frac{e^{-1}}{6}+\frac{2}{3}e^{-2}\right)$$

c.



Define $z^{1/3} = e^{\frac{1}{3}\log z}$

where $\log z := \log r + i\theta$, for $z = re^{i\theta}$, $0 \le \theta < 2\pi$.

Then $2\pi i \cdot \sum$ residues $\overset{\text{R.T.}}{=} \lim_{\epsilon\to 0} \int_{\gamma_{r,R,\epsilon}} \frac{z^{1/3}}{z^2+9z+8}\,dz$

$$= \left(1-e^{2\pi i/3}\right)\cdot \int_{-r}^{R} \frac{x^{1/3}}{x^2+9x+8}\,dx$$

$$\cancel{+} - \int_{\circlearrowleft} + \int_{\circlearrowleft}$$

as $r\to 0$ ↓ 0 ; ↓ 0 as $R\to\infty$

Estimates:
$$\left|\int_{\circlearrowleft} \frac{z^{1/3}}{z^2+9z+8}\,dz\right| \le 2\pi r \cdot C\cdot r^{1/3} \longrightarrow 0 \quad \text{as } r\to 0.$$

radius r

where $\left|\frac{z^{1/3}}{z^2+9z+8}\right| \le C_1 \cdot |z^{1/3}|$ for $|z|\le r_0$

some constant C_1, some $r_0 > 0$.

$$\left|\int_{\circlearrowleft} \frac{z^{1/3}}{z^2+9z+8}\,dz\right| \le 2\pi R\cdot C_2 \cdot \frac{R^{1/3}}{R^2} \longrightarrow 0 \quad \text{as } R\to\infty$$

radius R

where $\left|\frac{z^{1/3}}{z^2+9z+8}\right| \le C_2 \cdot \frac{|z^{1/3}|}{|z^2|}$ for $|z| \ge R_0$

some C_2, R_0.

$z^2+9z+8 = (z+1)(z+8) \Rightarrow$ simple poles at $z=-1,-8$.

Thus $(1-e^{2\pi i/3}) \cdot \int_0^\infty \frac{x^{1/3}}{x^2+9x+8}\,dx = 2\pi i \cdot \left(\frac{(-1)^{1/3}}{(2z+9)|_{z=-1}} + \frac{(-8)^{1/3}}{(2z+9)|_{z=-8}} \right)$

$= 2\pi i \left(e^{\pi i/3} \cdot \left(\frac{1}{7} + \frac{2}{-7} \right) \right)$

$= 2\pi i \cdot e^{\pi i/3} \cdot {-1/7}$

$\therefore \int_0^\infty \frac{x^{1/3}}{x^2+9x+8}\,dx = \frac{-2\pi i}{7} \cdot \frac{\left(\frac{1}{2}+\frac{\sqrt{3}}{2} i\right)}{\left(\frac{3}{2} - \frac{\sqrt{3}}{2} i\right)} = \frac{-2\pi i}{7} \cdot \frac{e^{\pi i/3}}{\sqrt{3} \cdot e^{-\pi i/6}}$

$= \frac{-2\pi i}{7\sqrt{3}} \cdot e^{\pi i/2} = \frac{-2\pi i \cdot i}{7\sqrt{3}} = \boxed{\frac{2\pi}{7\sqrt{3}}}$

6. $\int_\gamma z^n e^{2/z}\,dz$

$= \int_\gamma z^n \cdot \sum_{k=0}^\infty \frac{1}{k!} \left(\frac{2}{z}\right)^k dz$

convergence is uniform on γ

$= \sum_{k=0}^\infty \frac{2^k}{k!} \cdot \int_\gamma z^{n-k}\,dz$ $\qquad \int_\gamma z^m\,dz = \begin{cases} 0 & m \neq -1 \\ 2\pi i & m=-1 \end{cases}$

$= \frac{2^{n+1}}{(n+1)!}$

7. a. $f(z) = \frac{2z-1}{z(z-1)}$

$= \frac{2z-1}{z} \cdot \frac{-1}{1-z}$

$= -\left(\frac{2z-1}{z}\right) \cdot (1+z+z^2+\ldots)$

$= \left(\frac{1}{z} - 2\right) \cdot (1+z+z^2+\ldots)$

$= (z^{-1}+1+z+z^2+\ldots) - 2(1+z+z^2+\ldots)$

$= z^{-1} - 1 - z - z^2 - \ldots$

$$z^2-4z+3 = (z-1)(z-3)$$

b. $$f(z) = \frac{2z}{z^2-4z+3} = \frac{A}{z-1} + \frac{B}{z-3} \quad \text{(partial fractions)}$$

$$2z = A\cdot(z-3) + B\cdot(z-1)$$

$$A+B=2, \quad -3A-B=0, \quad B=3, \ A=-1$$

$$f(z) = \frac{-1}{z-1} + \frac{3}{z-3}$$

$$= \frac{1}{1-z} - \frac{1}{1-z/3}$$

$$= \sum_{n=0}^{\infty} z^n - \sum_{n=0}^{\infty} (z/3)^n = \sum_{n=0}^{\infty} (1-3^{-n})\cdot z^n$$

for $|z|<1$

$$f(z) = \frac{1}{z}\cdot\frac{-1}{1-1/z} - \frac{1}{1-z/3} = \frac{-1}{z}\cdot\sum_{n=-\infty}^{0} z^n - \sum_{n=0}^{\infty} 3^{-n}\cdot z^n$$

$$= -\sum_{n=-\infty}^{-1} z^{+n} - \sum_{n=0}^{\infty} 3^{-n}\cdot z^n \quad \text{for } 1<|z|<3.$$

$$f(z) = \frac{-1}{z}\cdot\frac{1}{1-1/z} + \frac{3}{z}\cdot\frac{1}{1-3/z} = -\sum_{n=-\infty}^{-1} z^{+n} + \sum_{n=-\infty}^{-1} 3^{-n} z^n$$

$$= \sum_{n=-\infty}^{-1} (3^{-n}-1)\, z^n \quad \text{for } |z|>3.$$

8. $$\tan(z) = \sum_{n=-\infty}^{\infty} a_n z^n \quad \text{valid in} \quad \pi/2 < |z| < 3\pi/2$$

$$a_n = \frac{1}{2\pi i}\int_\gamma \tan(z)\cdot z^{-(n+1)}\, dz$$

γ the circle centre the origin, radius R, $\pi/2 < R < 3\pi/2$, oriented ccw.

Now assume $n \leq -1$: So $-(n+1) \geq 0$

& $$a_n \overset{\text{R.T.}}{=} \sum \left(\text{residues of } \tan z \cdot z^{-(n+1)} \text{ inside } \gamma\right)$$

$$= \sum_{z_0 = \pm\pi/2} \operatorname{res}_{z_0}\left(\tan z\cdot z^{-(n+1)}\right) \qquad \tan z = \frac{\sin z}{\cos z}$$

$$= \sum_{z_0=\pm\pi/2} \frac{\sin z\cdot z^{-(n+1)}\big|_{z=z_0}}{(\cos z)'\big|_{z=z_0}} = -\left((\pi/2)^{-(n+1)} + (-\pi/2)^{-(n+1)}\right)$$

$$= \begin{cases} -2 \cdot \left(\frac{\pi}{2}\right)^{-(n+1)} & n \text{ odd} \\ 0 & n \text{ even.} \end{cases}$$

9.

$g(z) := f(z)/(\sin z)^3$ is hol. on $\mathbb{C} \setminus \{k\pi \mid k \in \mathbb{Z}\}$

& bounded.

$\therefore$ g has removable sings (by Riemann's thm)

& extends to $\tilde{g}$, hol on $\mathbb{C}$ & bdd.

$\therefore$ $\tilde{g}$ is constant (by Liouville's thm), $\tilde{g} = \lambda$

& $f(z) = \lambda \cdot (\sin z)^3$ $\square$.

10. $f : \mathbb{C} \to \mathbb{C}$ hol.

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ valid $\forall z \in \mathbb{C}$.

$g(z) := f(1/z) = \sum_{n=0}^{\infty} a_n \cdot z^{-n}$ valid $\forall z \neq 0$

f has an essential sing. at ∞ $\overset{\text{def.}}{\iff}$ g has essential sing at $z=0$

$\iff$ $a_n \neq 0$ for infinitely many n

$\iff$ f not polynomial. $\square$.

11. $f(z) = p(z)/q(z) = \dfrac{\overset{\neq 0}{c} \cdot \prod (z-\alpha_i)^{n_i}}{\prod (z-\beta_j)^{m_j}}$ $\quad n = \sum n_i = \deg p$, $m = \sum m_i = \deg q$.

WMA $\alpha_i \neq \beta_j$ $\forall i,j$.

At ∞ : $g(z) = f(1/z) = c \cdot z^{m-n} \dfrac{\prod (1-\alpha_i z)^{n_i}}{\prod (1-\beta_j z)^{m_i}}$

Finally, $\sum_{p \in \mathbb{C} \cup \{\infty\}\text{'s}} \operatorname{ord}_p(f) = \underset{(\text{zeros in } \mathbb{C})}{\sum n_i} - \underset{(\text{poles in } \mathbb{C})}{\sum m_j} + \underset{(\infty)}{m-n} = 0.$ $\square$

12. $$\int_\gamma \frac{1}{(z-3)(2+3z)^3(i-2z)^2}\,dz$$

$w = 1/z$ $$= -\int_\gamma \frac{1}{(w^{-1}-3)(2+3w^{-1})^3(i-2w^{-1})^2}\,\frac{-1}{w^2}\,dw$$ $\gamma = \{w \in \mathbb{C} \mid |w| = 1\}$, traversed ccw

orientation reversed

$$= \int_\gamma \frac{w^4}{(1-3w)(2w+3)^3(iw-2)^2}\,dw$$

poles: $w = \underline{1/3}, -3/2, -2i$

inside γ, simple.

$$= 2\pi i \sum (\text{residues at poles inside } \gamma)$$

$$= 2\pi i \cdot \left.\frac{w^4}{(2w+3)^3(iw-2)^2}\right|_{w=1/3} \Big/ -3$$

$$= 2\pi i \frac{1}{-3^5 \cdot (11/3)^3 \cdot (-2+i/3)^2} = \frac{-2\pi i}{3^2 \cdot 11^3 \cdot (-2+i/3)^2} = \frac{-2\pi i}{11^3 \cdot (-6+i)^2}$$

$$= \frac{-2\pi i\,(-6-i)^2}{11^3 \cdot 37^2} = \frac{-2\pi i\,(35+12i)}{11^3 \cdot 37^2} = \frac{2\pi}{11^3 37^2} \cdot (12-35i)$$