Math 621 Midterm review problems

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The midterm exam will be held Wednesday 3/7/18, 7:00PM–8:30PM, in LGRT 1322. The syllabus for the midterm exam is the following sections of Stein and Shakarchi: Chapter 1, Sections 1,2,3; Chapter 2, Sections 1,2,4; Chapter 3, Sections 1,2,3.

Justify your answers carefully.

(1) Let γ be the circle with center z_0 and radius R, oriented counterclockwise. Let $\Omega \subset \mathbb{C}$ be an open set such that γ and its interior are contained in Ω, and let $f\colon \Omega \to \mathbb{C}$ be a holomorphic function. Suppose $|f(z)| \leq M$ for all $z \in \gamma$.

   (a) For $n \geq 1$, give the precise statement of the Cauchy inequality for $f^{(n)}(z_0)$, the nth derivative of f evaluated at z_0.

   (b) Give examples to show that the Cauchy inequality is the best possible, that is, the bound on $f^{(n)}(z_0)$ cannot be improved for all functions f which are holomorphic inside and on γ.

(2) Let

$$P_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}.$$

Given a positive real number R, prove that P_n has no zeros in the disc with center the origin and radius R for all n sufficiently large.

(3) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function with domain $\mathbb{C}$ such that the real part of f is bounded above. Prove that f is constant.

(4) For each of the following functions, find all isolated singular points, classify them (into removable singularities, poles, essential singularities), and find the residues at all isolated singular points.

(a)
$$\frac{\sin z}{z(z-\pi/2)^2}$$

(b)
$$z^2 e^{1/(z+1)}$$

(c)
$$(\cot z)^2$$

(d)
$$\frac{z^{35}}{1-z^{16}}$$

(5) Evaluate the following integrals.

(a)
$$\int_{-\infty}^{\infty} \frac{1}{x^6+1} dx.$$

(b)
$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{x^4+5x^2+4} dx.$$

(c)
$$\int_0^{\infty} \frac{x^{1/3}}{x^2+9x+8} dx.$$

(6) Evaluate the integral
$$\int_\gamma z^n e^{2/z} dz,$$
where n is an integer and γ is a circle with center the origin, oriented counterclockwise.

(7) (a) Find the Laurent series of
$$f(z) = \frac{2z-1}{z(z-1)}$$

centered at $z = 0$ which is valid on the punctured disc $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$.

(b) Find all the Laurent series for $f(z) = \frac{2z}{z^2-4z+3}$ centered at $z = 0$ and specify for each the largest open set over which it represents the function.

(8) Consider the Laurent series $\tan(z) = \sum_{-\infty}^{\infty} a_n z^n$ which is valid in the annulus $\{z \in \mathbb{C} \mid \frac{\pi}{2} < |z| < \frac{3\pi}{2}\}$. Using contour integrals or otherwise, determine the coefficients a_n with index $-\infty < n \leq -1$.

(9) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function with domain $\mathbb{C}$ and suppose that $|f(z)| \leq |\sin z|^3$ for all $z \in \mathbb{C}$. Prove that $f(z) = \lambda(\sin z)^3$ for some $\lambda \in \mathbb{C}$.

(10) Let $f\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function with domain $\mathbb{C}$. Show that if f is not a polynomial then f has an essential singularity at infinity.

(11) Let f be a meromorphic function on $\mathbb{C} \cup \{\infty\}$. For $p \in \mathbb{C} \cup \{\infty\}$ define

$$\operatorname{ord}_p(f) = \begin{cases} n & \text{if } f \text{ has a zero of order } n \text{ at } p \\ -n & \text{if } f \text{ has a pole of order } n \text{ at } p \\ 0 & \text{otherwise.} \end{cases}$$

Prove that $\sum_{p \in \mathbb{C} \cup \{\infty\}} \operatorname{ord}_p(f) = 0$.

(12) Compute the integral

$$\int_\gamma \frac{1}{(z-3)(2+3z)^3(i-2z)^2} dz$$

where γ is the circle with center the origin and radius 1 oriented counter-clockwise.

[Hint: Consider the substitution $w = 1/z$.]