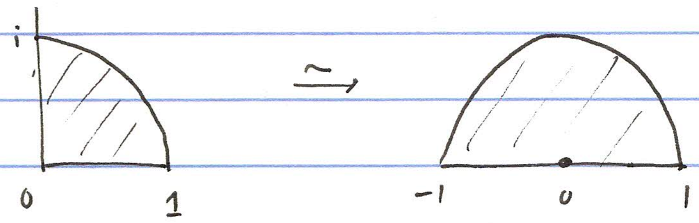
Sunday 5/6/18. MATH 621 Final review solutions.

1. By Rouché's theorem, since $|f(z)| < |z^3| = 1$ on ∂D,
$\#$ zeroes of $f(z) - z^3$ in D $=$ $\#$ zeroes of z^3 in D $= 3$
(note z^3 has a unique zero at $z=0$ of multiplicity 3).

2.

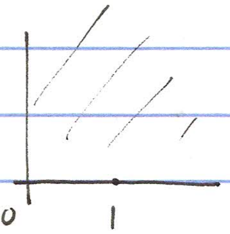
$\Omega =$  Ω_1

$z \longmapsto z^2$

$-1 \longmapsto 0$

$1 \longmapsto \infty$ M.T. $z \longmapsto \frac{z+1}{z-1} \Big/ \frac{1}{-1} = \frac{1+z}{1-z}$

$0 \longmapsto 1$

$\rightsquigarrow$ $\Omega_1 \xrightarrow[(*)]{\sim}$  $\Omega_2 \xrightarrow{\sim} H$

$z \longmapsto \frac{1+z}{1-z}$ $z \longmapsto z^2$

Combining, $\Omega \xrightarrow{\sim} H$ $z \longmapsto \left(\frac{1+z^2}{1-z^2}\right)^2$

$(*)$ Explanation: Möbius transformations $f(z) = \frac{az+b}{cz+d}$, $a,b,c,d \in \mathbb{C}$, $ad-bc \neq 0$

define holomorphic bijections $f: \mathbb{C}\cup\{\infty\} \xrightarrow{\sim} \mathbb{C}\cup\{\infty\}$ which map circles & lines to circles & lines, & preserve angles (including the orientation - clockwise vs. counterclockwise) because holomorphic w/ nonzero derivative at each point.

Now, since $-1, 1, 0 \mapsto 0, \infty, 1$ by construction of MT $f(z) = \frac{1+z}{1-z}$, we have $f(\mathbb{R}\cup\{\infty\}) = \mathbb{R}\cup\{\infty\}$, and $f([-1,1]) = \mathbb{R}_{\geq 0} \cup \{\infty\}$.

Since $-1 \mapsto 0$ & $1 \mapsto \infty$, the circle $C = \{z \mid |z| = 1\}$ maps to a line thru the origin, & since angles are preserved $f(C) = i\mathbb{R} \cup \{\infty\}$ & $f(\{z \in C \mid \operatorname{Im}(z) \geq 0\}) = i\mathbb{R}_{\geq 0} \cup \{\infty\}$ ($\mathbb{R}$ & C meet at angle $\pi/2$ at -1 $\Rightarrow$ $f(\mathbb{R})$ & $f(C)$ meet at angle $\pi/2$ at 0).

Now it follows that $f(\Omega_1) = \cancel{\mathbb{R}_{>0} + i\mathbb{R}_{>0}}\ \Omega_2$, the positive quadrant

(by the above & since f must map Ω_1, one of the connected components of $\mathbb{C} \cup \{\infty\} \setminus (\mathbb{R} \cup \{\infty\}) \cup C$, to one of the connected cpts of $\mathbb{C} \cup \{\infty\} \setminus (\mathbb{R} \cup \{\infty\}) \cup (i\mathbb{R} \cup \{\infty\}) = \mathbb{C} \setminus \mathbb{R} \cup i\mathbb{R}$).

3. $f : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ hol.

Recall if $f : \{z \in \mathbb{C} \mid R_1 < |z| < R_2\} \longrightarrow \mathbb{C}$ hol then f has a Laurent series expansion $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ valid on its domain.

In our case ($R_1 = 0$, $R_2 = \infty$) we get $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ $\forall z \in \mathbb{C}^*$

f cannot have an essential singularity at $z = 0$ or $z = \infty$ (by Casorati–Weierstrass theorem & open mapping theorem this would contradict injectivity of f).

Thus $f(z) = a_k z^k + a_{k+1} z^{k+1} + \dots + a_l z^l$ (finite sum), some $k, l \in \mathbb{Z}$, $k < l$, $a_k, a_l \neq 0$.

$$f(z) = \cancel{a_k z^k}\ z^k (a_k + a_{k+1} z + \dots + a_l z^{l-k})$$

$f(z) \neq 0$ $\forall z \in \mathbb{C}^*$ & Fundamental theorem of algebra $\Rightarrow$ $l = k$, $f(z) = a_k z^k$.

f injective $\Rightarrow$ $k = \pm 1$. $\square$

4. Observe that $g(z) := -f(-z)$ is another hol. bijection from D to S.

And $g(0) = f(0) = 0$.

$g'(0) = -f'(0) \cdot -1 = f'(0)$ by the chain rule

Now it follows that $f = g$:-

Consider $g^{-1} \circ f : D \longrightarrow D$

$g^{-1} \circ f\,(0) = 0$, $(g^{-1} \circ f)'(0) = g'(0)^{-1} \cdot f'(0) = 1$

Schwarz Lemma $\Rightarrow$ $g^{-1} \circ f$ rotation $z \mapsto e^{i\theta} \cdot z$

where $e^{i\theta} = (g^{-1} \circ f)'(0) = 1$

i.e. $g^{-1} \circ f = id$, $f = g$ $\square$.

5. $f(z) = z^7 - 5z^3 + 12$.

$|z| = 2$:- $|z^7| = 128$, $|-5z^3 + 12| \leq 5|z|^3 + 12 = 52 < 128$.

$\therefore$ Rouché's thm $\Rightarrow$ # zeros of f in $|z| < 2$

$= \#$ zeros of z^7 in $|z| < 2$ $= 7$

$|z| = 1$:- $|z^7 - 5z^3| \leq |z|^7 + 5|z|^3 = 6 < |12| = 12$.

$\therefore$ Rouché's thm $\Rightarrow$ # zeros of f in $|z| < 1$

$= \#$ zeros of 12 in $|z| < 1$ $= 0$.

$\therefore$ # zeros of f in $A = \{ z \mid 1 < |z| < 2 \}$ $= 7 - 0 = 7$.

6. $f(z) = \tan z = \dfrac{\sin z}{\cos z} = \dfrac{\frac{e^{iz} - e^{-iz}}{2i}}{\frac{e^{iz} + e^{-iz}}{2}} = \dfrac{1}{i} \cdot \dfrac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$

$= \dfrac{1}{i} \dfrac{e^{2iz} - 1}{e^{2iz} + 1}$.

$S = \{ z \in \mathbb{C} \mid -\pi/4 < Re(z) < \pi/4 \} \xrightarrow{\sim} ?$

$z \longmapsto e^{2iz}$

$\{z \in \mathbb{C} \mid -\pi/4 < Re(z) < \pi/4\} \xrightarrow{\sim} \{z \in \mathbb{C} \mid -\pi/2 < Im(z) < \pi/2\}$

$z \longmapsto 2i \cdot z$

$\xrightarrow{\sim} \{z \in \mathbb{C} \mid -\pi/2 < \arg(z) < \pi/2\} = \{z \in \mathbb{C} \mid Re(z) > 0\}$

$z \longmapsto e^z.$

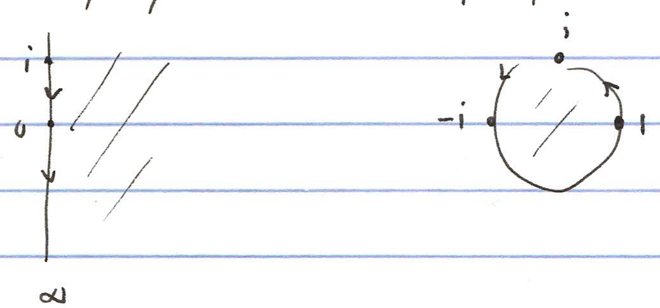
$\xrightarrow{\sim}$?

$z \longmapsto \frac{1}{i} \frac{z-1}{z+1}$

$\therefore$ $0 \longmapsto i$, $\infty \longmapsto -i$, $i \longmapsto 1$.

So $i\mathbb{R} \longrightarrow C = \{z \mid |z| = 1\}$

Also $i, 0, \infty \longmapsto 1, i, -i$ are in the same order on the boundaries of the two regions $\{z \in \mathbb{C} / Re(z) > 0\}$ & D (as we traverse the boundary w/ the region on our left).



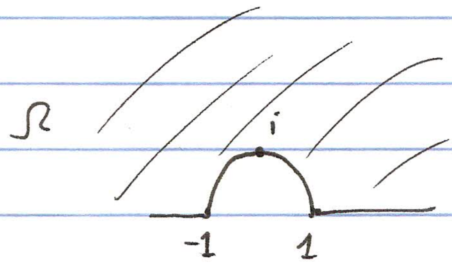
Now it follows that $\{z \in \mathbb{C} \mid Re(z) > 0\} \xrightarrow{\sim} D$.

(Alternatively, $\{z \in \mathbb{C} \mid Re(z) > 0\}$ is sent to one of the connected cpts of $\mathbb{C} \cup \{\infty\} \setminus C$, to determine which one we can compute the image of one point in $\{z \in \mathbb{C} \mid Re(z) > 0\}$).

Combining, we see $f(z) = \tan z = \frac{1}{i} \frac{e^{2iz} - 1}{e^{2iz} + 1} : \{z \in \mathbb{C} \mid -\pi/4 < Re(z) < \pi/4\} \xrightarrow{\sim} D.$

$\square$.

7.



M.T. $-1 \longmapsto 0$

$1 \longmapsto \infty$

$i \longmapsto 1$

$f: z \longmapsto \frac{z+1}{z-1} \Big/ \frac{i+1}{i-1} = i \cdot \left(\frac{z+1}{z-1}\right)$

By construction

$$f(C) = \mathbb{R} \cup \{\infty\} \qquad (C = \{z \mid |z| = 1\})$$

$$f(\{z \in \mathbb{C} \mid \mathrm{Im}(z) \geq 0\}) = \mathbb{R}_{\geq 0} \cup \{\infty\}$$

Now it follows by preservation of angles as in Q2 that

$$f(\Omega) = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0 \ \& \ \mathrm{Im}(z) > 0\}$$

Finally $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0 \ \& \ \mathrm{Im}(z) > 0\} \xrightarrow{\sim} H$

$$z \longmapsto z^2$$

So, combining, $\Omega \xrightarrow{\sim} H$

$$z \longmapsto \left(i\left(\frac{z+1}{z-1}\right)\right)^2 = -\left(\frac{z+1}{z-1}\right)^2.$$

8. $$f(z) = \sum_{n=-\infty}^{n=\infty} \frac{1}{(z-n)^2}$$

For $|z| \leq R$ $\quad \left|\frac{1}{(z-n)^2}\right| \Big/ \frac{1}{n^2} = \frac{n^2}{|z-n|^2} = \frac{1}{|1-z/n|^2} < \frac{1}{(1-R/n)^2} \to 1$ as $n \to \infty$.

Now, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent,

see $f_N(z) = \sum_{n=-N}^{N} 1/(z-n)^2$ is uniformly convergent to $f(z)$ on compact sets in $\mathbb{C} \setminus \mathbb{Z}$.

Thus ~~far~~ $f(z)$ is holomorphic on $\mathbb{C} \setminus \mathbb{Z}$.

Similarly, at a point $n \in \mathbb{Z} \subset \mathbb{C}$, removing the term $1/(z-n)^2$,

see $f(z) = \frac{1}{(z-n)^2} + g(z)$, $g(z)$ hol at $n \in \mathbb{Z} \subset \mathbb{C}$.

Thus f is meromorphic on $\mathbb{C}$ w/ double pole at each $n \in \mathbb{Z}$.

even: $$f(-z) = \sum_{n \in \mathbb{Z}} \frac{1}{(-z-n)^2} = \sum_{n \in \mathbb{Z}} \frac{1}{(z+n)^2} \overset{m=-n}{=} \sum_{m \in \mathbb{Z}} \frac{1}{(z-m)^2} = f(z)$$

Periodic: $f(z+1) = \sum_{n\in\mathbb{Z}} \frac{1}{(z+1-n)^2} = \sum_{n\in\mathbb{Z}} \frac{1}{(z-(n-1))^2}$

$$\underset{m=n-1}{=} \sum_{m\in\mathbb{Z}} \frac{1}{(z-m)^2} = f(z)$$

b. $\sin(\pi z)$ has zeroes at $z \in \mathbb{Z} \subset \mathbb{C}$ (& nowhere else).

$\Rightarrow \left(\frac{\pi}{\sin \pi z}\right)^2$ meromorphic on $\mathbb{C}$, hol. on $\mathbb{C}\setminus\mathbb{Z}$.

Compute principal part of Laurent expansion at $n \in \mathbb{Z}$, check

$= \frac{1}{(z-n)^2} + 0\cdot\frac{1}{(z-n)} + \text{hol} \Rightarrow f(z) - \left(\frac{\pi}{\sin \pi z}\right)^2$ has removable sing. at $z=n$.

By periodicity $(\sin \pi(z+1) = \sin \pi z)$, enough to check $n=0$.

$$\left(\frac{\pi}{\sin \pi z}\right)^2 = \left(\frac{\pi}{\pi z - \frac{\pi^3 z^3}{3!} + \dots}\right)^2 = \frac{1}{z^2}\left(\frac{1}{1-\frac{\pi^2 z^2}{6}+\dots}\right)$$

$$= \frac{1}{z^2}\left(1 + \frac{\pi^2 z^2}{6} + \dots\right) = \frac{1}{z^2} + \text{hol.}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

9. $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$

if $|z| \le r < 1$ ~~$\left|\frac{z^n}{1+z^{2n}}\right| / |z^n| = \frac{1}{|1+z^{2n}|} \le \frac{1}{1-r^{2n}} \to 1$ as $n\to\infty$.~~

$$\left|\frac{z^n}{1+z^{2n}}\right| / r^n \le \frac{1}{1-r^{2n}} \to 1 \text{ as } n \to \infty$$

& $\sum_{n=1}^{\infty} r^n = 1/_{1-r}$ convergent $\Rightarrow f_N(z) = \sum_{n=1}^{N} z^n/_{1+z^{2n}}$

converges uniformly on compact sets in $|z|<1$ to $f(z)$.

Thus f hol on $|z|<1$.

Note $f(z^{-1}) = \sum_{n=1}^{\infty} \frac{z^{-n}}{1+z^{-2n}} = \sum_{n=1}^{\infty} \frac{z^{n}}{z^{2n}+1} = f(z)$

Thus f hol on $|z|>1$.

Finally if $|z|=1$, $\left|\frac{z^n}{1+z^{2n}}\right| \geqslant \frac{|z|^n}{1+|z|^{2n}} = \frac{1}{2} \not\to 0$ as $n \to \infty$.

So series is divergent.

10. $g(z) := f(z) - f(-z)$

$g : D \longrightarrow \mathbb{C}$, $g(D) \subset d \cdot D$

by definition of diameter d of $f(D)$ & maximum principle

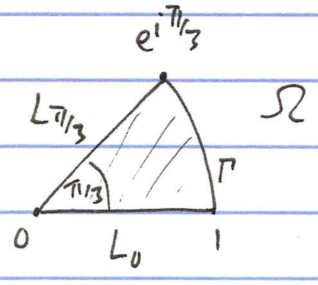
Also $g(0)=0$.

Apply Schwarz Lemma to $h = 1/d \cdot g$

$\Rightarrow |h'(0)| \leq 1$, i.e., $|g'(0)| \leqslant d$

$g'(0) = 2f'(0)$ by chain rule, so $|f'(0)| \leq 1/2\, d$. $\square$

11.



f M.T., $f(1) = i$, $f(e^{i\pi/3}) = 0$, $f(\Gamma) \subset \mathbb{R}i$,
(containing the arc Γ) $f(L_{\pi/3}) \subset \mathbb{R}$. ?

Necessarily, the unit circle C is sent to $\mathbb{R}i$ & the line L containing the line segment $L_{\pi/3}$ is sent to $\mathbb{R}$.

Thus one of the intersection pts $C \cap L$ is sent to ∞.

" $\{\pm e^{i\pi/3}\}$

By assumption $e^{i\pi/3} \mapsto 0$, so $e^{-i\pi/3} \mapsto \infty$.

Now compute: $$f(z) = i \cdot \left(\frac{z - e^{i\pi/3}}{z - e^{-i\pi/3}} \right) \Big/ \left(\frac{1 - e^{i\pi/3}}{1 - e^{-i\pi/3}} \right) \qquad (\dagger)$$

Check this works:

By construction $1, e^{i\pi/3}, e^{-i\pi/3} \mapsto i, 0, \infty$

$\Rightarrow f(C) = i\mathbb{R} \cup \{\infty\}$

$f(L)$ is a line (because $L \ni e^{-i\pi/3} \mapsto \infty$)

& by preservation of angles, $f(L) = \mathbb{R} \cup \{\infty\}$

(C & L make angle $\pi/2$ at $e^{i\pi/3}$ $\Rightarrow$ $f(C)$ & $f(L)$ make angle $\pi/2$ at 0).

Simplify formula $(\dagger)$:-

$$f(z) = i \cdot \left(\frac{z - e^{i\pi/3}}{z - e^{-i\pi/3}} \right) \Big/ -e^{i\pi/3}$$

$$= -e^{i\pi/6} \cdot \left(\frac{z - e^{i\pi/3}}{z - e^{-i\pi/3}} \right).$$

12.

$$f(z) = z^n + a_{n-1} z^{n-1} + \dots + a_0$$

$|z| = R = A+1$. :- $|a_{n-1} z^{n-1} + \dots + a_1 z + a_0| \leq A \cdot (|z|^{n-1} + \dots + |z| + 1)$

$= (R-1)(R^{n-1} + \dots + R + 1)$

$= R^n - 1 < R^n = |z|^n$

Rouché's thm $\Rightarrow$ # zeroes of f in $|z| < R$

$=$ # zeroes of z^n in $|z| < R$ $= n$. $\square$

13. $f(z) = 2z^2 + \sin z$.

$$|\sin z| = \left| \frac{e^{iz} - e^{-iz}}{2i} \right| \leq \frac{|e^{iz}| + |e^{-iz}|}{2} = \frac{e^{-y} + e^{y}}{2} = \cosh y$$

($e^{iz} = e^{i(x+iy)} = e^{-y+ix} \Rightarrow |e^{iz}| = e^{-y}$, sim. $|e^{-iz}| = e^{y}$)

Thus, for $|z| = 1$

$$|\sin z| \leq \cosh 1 = \frac{e+e^{-1}}{2} < \frac{3+1}{2} = 2 = |2z^2|.$$

Rouché's thm $\Rightarrow$ # zeros of $f(z)$ in D
= # zeros of $2z^2$ in D $= 2$.

Notice $z=0$ is a simple zeo (simple because $f'(0) \neq 0$)
Thus there is one additional zeo, necessarily simple.

14. a. Yes : a holomorphic fn $f: D \to \mathbb{C}$ always has a primitive.
because D is simply connected

b. No : $\displaystyle\int_\gamma f\,dz \underset{R.T.}{=} 2\pi i \cdot \sum_{\alpha = \pm i} \mathrm{Res}_{z=\alpha} f(z)$

$\gamma = \{z \in \mathbb{C} \mid |z| = 2\} \subset \Omega_2$, oriented ccw.

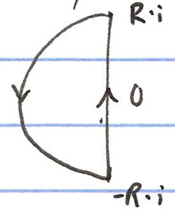
$$f(z) = \frac{g(z)}{(z+i)(z-i)} \longrightarrow = 2\pi i \left(\frac{g(i)}{(i+i)} + \frac{g(-i)}{(-i-i)} \right)$$

$$= \pi \cdot (g(i) - g(-i)) \neq 0$$

$\Rightarrow$ f does not have a primitive on Ω_2.

15. $f: \mathbb{C} \to \mathbb{C}$, $f(z) = z^5 + e^z + 4$, $\Omega = \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$.

Let γ_R be the curve :



Then, for $z \in \gamma_R$, & $R >> 0$ have

$$|z^5 + 4| > \left(|e^z| = e^x \leq 1 \right) \quad :-$$

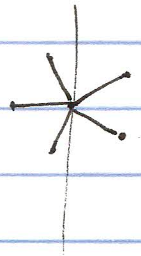
For $|z| = R$, $|z^5+4| \geq R^5 - 4$

For $z \in i\mathbb{R}$, $|z^5+4| = |i\,t^5 + 4| \geq 4$
$z = it$

Thus # zeros of inside γ_R = # zeros z^5+4 inside γ_R $= 3$:-

$z^5+4=0 \iff z^5=-4=4.e^{i\pi}$

$\iff z = \sqrt[5]{4}\,.\,e^{i\pi/5+2\pi/5\cdot k} \qquad k=0,1,..,4$



$R\to\infty \implies$ # zeros of in Ω $=3$. $\square$.

16. Use RMT: $\exists$ hol. bijection $G:\Omega\to D$

(OR $\Omega=\mathbb{C}$, in which case can use translation $F:\mathbb{C}\to\mathbb{C}$, $F(z)=z+(z_2-z_1)$)

& Blaschke factors $\psi_\alpha : D \xrightarrow{\simeq} D$

$$z \longmapsto \frac{\alpha - z}{1-\bar{\alpha}z}$$

$$\alpha \longmapsto 0 \quad , \quad 0\longmapsto \alpha$$

Now

$$\Omega \xrightarrow{G} D \xrightarrow[\simeq]{\psi_{G(z_1)}} D \xrightarrow[\simeq]{\psi_{G(z_2)}} D \xrightarrow[\simeq]{G^{-1}} \Omega$$

$$z_1 \longmapsto G(z_1) \longmapsto 0 \longmapsto G(z_2) \longmapsto z_2$$

this composition is the desired $F:\Omega\xrightarrow{\simeq}\Omega$, $z_1\mapsto z_2$. $\square$.

17. $\Omega=\{z\in\mathbb{C} \mid |z-1|>2\}$. $f:\Omega\to\mathbb{C}$, $f(z)=\dfrac{\cos(\pi z)}{z\cdot(z-2)}$

$\mathbb{C}\setminus\Omega = \{z\in\mathbb{C} \mid |z-1|\le 2\}$ connected. $f(z)$ is meromorphic on $\mathbb{C}$ w/ poles at $z=0$ & $z=2$.

So, for any loop γ in Ω

$$\int_\gamma f\,dz \underset{R.T.}{=} 2\pi i\left(\left(\operatorname{Res}_{z=0} f(z)\right)\cdot n(\gamma,0) + \left(\operatorname{Res}_{z=2} f(z)\right)\cdot n(\gamma,2)\right)$$

$$= 2\pi i\cdot n\cdot\left(\operatorname{Res}_{z=0} f(z) + \operatorname{Res}_{z=2} f(z)\right)$$

where $n=n(\gamma,0)=n(\gamma,2)$:- winding numbers are equal b/c 0 & 2 lie in same $\to$

connected component of $\mathbb{C} \setminus \gamma$.

Now compute $\quad \operatorname{Res}_{z=0} f(z) = \dfrac{\cos(\pi \cdot 0)}{0-2} = -1/2$

$$\operatorname{Res}_{z=2} f(z) = \frac{\cos(\pi \cdot 2)}{2} = +1/2$$

So $\quad \displaystyle\int_\gamma f \, dz = 0 \quad \forall$ loops $\gamma \subset \Omega$. $\quad (\dagger)$

Now, can define $\quad \displaystyle g(z) := \int_{\gamma_z} f(w) \, dw$

γ_z a path from z_0 to z in Ω $\quad$ (fix basepoint $z_0 \in \Omega$)

Then g is well defined (does not depend on the choice of path γ_z by $\dagger$)

And $g' = f$ by usual argument of fundamental thm of calculus:-

$$g'(z) = \lim_{h \to 0} \frac{1}{h} \int_{[z, z+h]} f(w) \, dw$$

$$|g'(z) - f(z)| = \lim_{h \to 0} \left| \frac{1}{h} \int_{[z,z+h]} f(w) \, dw - f(z) \right|$$

$$= \lim_{h \to 0} \left| \frac{1}{h} \int_{[z,z+h]} (f(w) - f(z)) \, dw \right|$$

$$\leq \lim_{h \to 0} \frac{1}{\cancel{|h|}} \cdot \cancel{|h|} \cdot \sup_{w \in [z,z+h]} |f(w) - f(z)| = 0$$

f cts.

18. a $\quad f(z) = z^4 - 6z + 3 \qquad A = \{ z \in \mathbb{C} \mid 1 < |z| < 2 \}$

$|z| = 2$: $\quad (|z^4| = 16) > (|-6z+3| \leq 6|z| + 3 = 15)$ ✓

So # zeros f in $|z| < 2$ $=$ # zeros z^4 in $|z| < 2 = 4$

$|z| = 1$ $\quad |6z| = 6 > (|z^4 + 3| \leq |z|^4 + 3 = 4)$

So # zeros f in $|z| < 1$ $=$ # zeros $\cancel{z^4+3}$ $6z$ in $|z| < 1 = 1$

$\therefore$ # zeros f in $A = 4 - 1 = 3$.

b. Simple zeros: RTP $f'(\alpha) \neq 0$ for each zero α of f.

$f'(z) = 4z^3 - 6$

Just check f & f' have no common zeros:

$z^4 - 6z + 3 = 0$

& $4z^3 - 6 = 0$

$\Rightarrow z^3 = 6/4 = 3/2$, $0 = z^4 - 6z + 3 = 3/2 \cdot z - 6z + 3$
$= -9/2 \cdot z + 3$,

$\Rightarrow z = -3/-9/2 = 2/3$ # $z^3 \neq 3/2$. □.

19. a. $p(z) = z^5 - z^4 + 2z^3 - 3z^2 - 5$

$|z| = 3$ $\quad |z|^5 = 243 > \left(|-z^4 + 2z^3 - 3z^2 - 5| \leq |z|^4 + 2|z|^3 + 3|z|^2 + 5 = 81 + 2 \cdot 27 + 3 \cdot 9 + 5 = 167. \right)$ ✓

∴ Rouché $\Rightarrow$ # zeros of p in $|z| < 3$
$=$ # zeros of z^5 in $|z| < 3$ $= 5$.

b. By part a, all poles of ~~denominator~~ integrand contained in $|z| < 3$.
So, to compute integral, make substitution $w = 1/z$
(then will have single pole (of same order) at $w = 0$ inside contour in w plane, corresponding to pole of original integral at $z = \infty$, & R.T. will be easy to apply):-

$$\int_C \frac{z^4 - 2z^2 + z - 3}{z^5 - z^4 + 2z^3 - 3z^2 - 5} dz = -\int_{|w| = 1/3} \frac{w^{-4} - 2w^{-2} + w^{-1} - 3}{w^{-5} - w^{-4} + 2w^{-3} - 3w^{-2} - 5} \, -w^{-2} dw$$

$|w| = 1/3$ oriented cw

(reverse orientation of contour)

$$= \int_{|w| = 1/3} \underbrace{\frac{1 - 2w^2 + w^3 - 3w^4}{w - w^2 + 2w^3 - 3w^4 - 5w^6}} dw$$

$$\underset{\text{R.T.}}{=} 2\pi i \cdot \operatorname{Res}_{w=0} \left(\quad \right) \overset{\text{simple pole}}{=} 2\pi i \left(\frac{1 - 2w^2 + w^3 - 3w^4}{1 - w + 2w^2 - 3w^3 - 5w^5} \right)\Bigg|_{w=0} = 2\pi i$$

20. Suppose $|f(z)-\alpha| > \epsilon > 0 \quad \forall z \in \mathbb{C}$.

Then consider composition

$$\mathbb{C} \xrightarrow{f} \mathbb{C}\setminus\{\alpha\} \longrightarrow \mathbb{C}$$
$$z \longmapsto \frac{1}{z-\alpha}$$

$$g(z) = \frac{1}{f(z)-\alpha}, \quad g : \mathbb{C} \longrightarrow \{z \in \mathbb{C} \mid |z| < 1/\epsilon\}$$

Liouville $\Rightarrow$ g constant $\Rightarrow$ f constant $\#$. $\square$.

21. Schwarz Lemma: $\Rightarrow$ $|f(z)| \leq |z| \quad \forall z \in D$

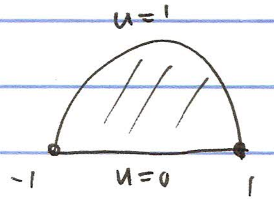
$$g(z) := \sum_{n=0}^{\infty} f(z^n)$$

For $|z| \leq r < 1$ $\quad |f(z^n)| \leq |z^n| \leq r^n$

& $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ convergent

$\Rightarrow$ series converges uniformly on compact sets, thus g is hol.

22. $\Omega = \{z \in \mathbb{C} \mid |z| < 1, \operatorname{Im}(z) > 0\}$



u harmonic on Ω, cts on $\overline{\Omega} \setminus \{-1, 1\}$, bdd?

Use conformal mapping: As in Q2, M.T. $f(z) = \frac{1+z}{1-z}$ sends Ω

to positive quadrant Q, w/ $[-1,1] \longrightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$

$\{z \in \mathbb{C} \mid |z|=1 \ \& \ \operatorname{Im}(z) \geq 0\} \longrightarrow i\mathbb{R}_{\geq 0} \cup \{\infty\}$

Have harmonic function $\tilde{u} : Q \longrightarrow \mathbb{R}$, $\tilde{u}(z) = \frac{2}{\pi}\arg(z)$

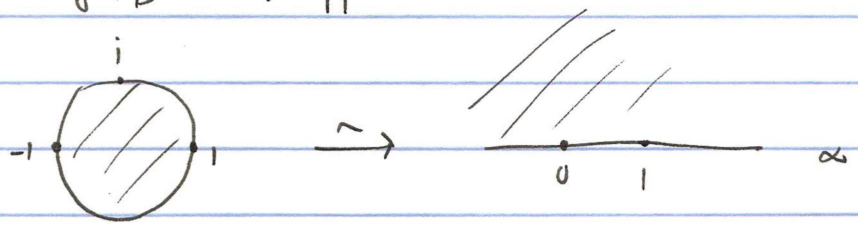
w/ desired boundary values (& bdd). Now $u(z) = \tilde{u}(f(z))$

$$= \frac{2}{\pi}\arg\left(\frac{1+z}{1-z}\right) \quad \square$$

23 a. $u_1 = y = \mathrm{Im}(z)$

b.

$f : D \xrightarrow{\sim} H$



$1 \mapsto 0, \quad i \mapsto 1, \quad -1 \mapsto \infty$

$$f(z) = \frac{z-1}{z+1} \Big/ \frac{i-1}{i+1} = i \cdot \left(\frac{1-z}{1+z}\right)$$

$u_2 = u_1 \circ f$, i.e. $u_2(z) = \mathrm{Im}\left(i \cdot \left(\frac{1-z}{1+z}\right)\right)$

$$= \mathrm{Re}\left(\frac{1-z}{1+z}\right)$$

$$= \mathrm{Re}\left(\frac{(1-x)-iy}{(1+x)+iy}\right) = \frac{(1-x)(1+x) - y^2}{(1+x)^2+y^2}$$

$$= \frac{1-x^2-y^2}{(1+x)^2+y^2}. \qquad (z = x+iy).$$