Math 621 Final review problems

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The final exam will be held Tuesday 5/8/18, 8:00AM–10:00AM, in LGRT 1322.

Justify your answers carefully.

(1) Let $\Omega \subset \mathbb{C}$ be an open set containing the closure $\bar{D}$ of the unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and $f \colon \Omega \to \mathbb{C}$ a holomorphic function such that $|f(z)| < 1$ for z on the unit circle ∂D. Show that the equation $f(z) = z^3$ has 3 solutions in D counting multiplicities.

(2) Let
$$\Omega = \{z \in \mathbb{C} \mid |z| < 1, \, \mathrm{Re}(z) > 0, \, \mathrm{Im}(z) > 0\}$$
be the portion of the unit disc contained in the positive quadrant and
$$H = \{z \in \mathbb{C} \mid \, \mathrm{Im}(z) > 0\}$$
be the upper half plane. Determine explicitly a holomorphic bijection $F \colon \Omega \to H$.

(3) Show that if $f \colon \mathbb{C}^\times \to \mathbb{C}^\times$ is a holomorphic bijection then either $f(z) = cz$ or $f(z) = cz^{-1}$, where $c \in \mathbb{C}^\times$ is a nonzero constant. (Here $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$.)

(4) Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ be the unit disc and
$$S = \{z \in \mathbb{C} \mid |\,\mathrm{Re}(z)| < 1, \, |\,\mathrm{Im}(z)| < 1\}$$
be the interior of the unit square. Suppose $f \colon D \to S$ is a holomorphic bijection such that $f(0) = 0$. Prove that $f(-z) = -f(z)$.

(5) How many zeroes does the polynomial $f(z) = z^7 - 5z^3 + 12$ have in the annulus $A = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$ (counting multiplicities)?

(6) Show that $f(z) = \tan(z)$ defines a holomorphic bijection from the strip $S = \{z \in \mathbb{C} \mid -\pi/4 < \mathrm{Re}(z) < \pi/4\}$ to the unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$.

(7) Let

$$\Omega = \{z \in \mathbb{C} \mid |z| > 1,\ \mathrm{Im}(z) > 0\},$$

the portion of the upper half plane lying outside the closure of the unit disc, and let H be the upper half plane. Determine a holomorphic bijection $F\colon \Omega \to H$.

(8) Let

$$f(z) = \sum_{n=-\infty}^{n=\infty} \frac{1}{(z-n)^2}$$

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(a) Show that the series defines a meromorphic function $f\colon \mathbb{C} \to \mathbb{C}$, with a double pole at each $n \in \mathbb{Z}$, such that f is even and satisfies $f(z+1) = f(z)$.

(b) Show that $g(z) := f(z) - (\pi/\sin(\pi z))^2$ is holomorphic on $\mathbb{C} \setminus \mathbb{Z}$ and has a removable singularity at each $n \in \mathbb{Z}$, that is, it extends to a holomorphic function $\tilde{g}\colon \mathbb{C} \to \mathbb{C}$.

(9) Let $f(z) = \sum_{n=1}^{\infty} \frac{z^n}{1+z^{2n}}$. Show that the series converges iff $|z| \neq 1$ and defines a holomorphic function on the open set $\Omega = \{z \in \mathbb{C} \mid |z| \neq 1\}$.

(10) Let $D \subset \mathbb{C}$ be the unit disc and $f\colon D \to \mathbb{C}$ a holomorphic function. Let $d = \sup\{|f(z_1) - f(z_2)| \mid z_1, z_2 \in D\}$ be the diameter of the image $f(D)$ of D under f. Prove that $|f'(0)| \leq \frac{1}{2}d$.

[Hint: Consider $g(z) := f(z) - f(-z)$.]

(11) Let Ω be the portion of the unit disc given in polar coordinates by

$$\Omega = \{z = re^{i\theta} \mid 0 < r < 1,\ 0 < \theta < \pi/3\}.$$

The boundary of Ω consists of the line segment L_0 from 0 to 1, the line segment $L_{\pi/3}$ from 0 to $e^{\pi i/3}$, and a curve Γ on the unit circle. Prove that there exists a unique Möbius transformation f satisfying $f(1) = i$, $f(e^{\pi i/3}) = 0$, f maps Γ into the imaginary line $\mathbb{R}i$, and f maps $L_{\pi/3}$ into the real axis. Give an explicit, simple formula for $f(z)$. Justify your answer.

[Hint: Find $f^{-1}(\infty)$ first.]

(12) Prove the following more precise version of the fundamental theorem of algebra. Let

$$f(z) = z^n + a_{n-1}z^{n-1} + \quad + a_1 z + a_0$$

be a monic polynomial of degree n with complex coefficients. Let A be the maximum of $|a_0|,|a_1|,\ldots,|a_{n-1}|$. Then f has n roots (counting multiplicities) in the open disc with center 0 and radius $R = A + 1$.

(13) Determine the number of zeroes of the function $f(z) = 2z^2 + \sin z$ in the open unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and show that all the zeroes are simple.

(14) Let $g \colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that $g(i) \neq g(-i)$. Define domains

$$\Omega_1 = \{z \in \mathbb{C} \mid |z| < 1\}$$

and

$$\Omega_2 = \{z \in \mathbb{C} \mid |z| > 1\}.$$

(a) Does there exist a holomorphic function $f : \Omega_1 \to \mathbb{C}$ such that $f'(z) = \frac{g(z)}{z^2+1}$?

(b) Does there exist a holomorphic function $f : \Omega_2 \to \mathbb{C}$ such that $f'(z) = \frac{g(z)}{z^2+1}$?

(15) Consider the function $f \colon \mathbb{C} \to \mathbb{C}$, $f(z) = z^5 + e^z + 4$. Let

$$\Omega = \{z \in \mathbb{C} \mid \operatorname{Re}(z) < 0\}$$

be the left half plane. Show that f has exactly 3 zeroes in Ω (counting multiplicities).

(16) Let Ω be a simply connected open subset of the complex plane. Show that for any two points $z_1, z_2 \in \Omega$ there exists a holomorphic bijection $F \colon \Omega \to \Omega$ such that $F(z_1) = z_2$.

(17) Let $\Omega = \{z \in \mathbb{C} \mid |z-1| > 2\}$ and consider the holomorphic function $f : \Omega \to \mathbb{C}$, $f(z) = \frac{\cos(\pi z)}{z(z-2)}$. Show carefully that there exists a holomorphic function $g : \Omega \to \mathbb{C}$ such that $g' = f$.

(18) (a) Find the number of solutions (counting multiplicities) of the equation $z^4 - 6z + 3 = 0$ in the annulus $A = \{z \in \mathbb{C} \mid 1 < |z| < 2\}$.

(b) Show that the multiplicity of each solution in part (a) is equal to 1.

(19) (a) Determine the number of zeroes of the polynomial

$$p(z) = z^5 - z^4 + 2z^3 - 3z^2 - 5$$

in the disc $\{z \in \mathbb{C} \mid |z| < 3\}$.

(b) Evaluate the integral

$$\int_C \frac{z^4 - 2z^2 + z - 3}{z^5 - z^4 + 2z^3 - 3z^2 - 5} dz,$$

where C is the boundary of the disc from part (a) with the counterclockwise orientation.

(20) Let $f : \mathbb{C} \to \mathbb{C}$ be a non-constant holomorphic function. Show that the image of f is dense in the complex plane.

(21) Let f be a holomorphic function on the unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$, such that $|f(z)| < 1$ for all $z \in D$ and $f(0) = 0$. Show that the series $g(z) := \sum_{n=0}^{\infty} f(z^n)$ defines a holomorphic function on D.

(22) Find a function u, harmonic and bounded on the domain

$$\Omega = \{z \in \mathbb{C} \mid |z| < 1, \operatorname{Im}(z) > 0\},$$

with the following boundary values:

(a) $u = 0$ on $\{z \in \mathbb{C} \mid |z| < 1, \operatorname{Im}(z) = 0\}$, and

(b) $u = 1$ on $\{z \in \mathbb{C} \mid |z| = 1, \operatorname{Im}(z) > 0\}$.

(23) (a) Let H be the upper half plane. Describe a function $u_1 : \bar{H} \to \mathbb{R}$ such that u is harmonic on H, continuous on $\bar{H}$, and $u|_{\partial H} = 0$.

(b) Let D be the unit disc. Using part (a) or otherwise, describe a function $u_2 \colon \bar{D} \setminus \{-1\} \to \mathbb{R}$ such that u is harmonic on D, continuous on $\bar{D} \setminus \{-1\}$, and $u|_{\partial D \setminus \{-1\}} = 0$.

[Remark: These functions do not contradict the uniqueness statement of the generalized Dirichlet problem because they are *not* bounded.]