Tuesday 10/22/19.

MATH 611 MIDTERM REVIEW SOLUTIONS

1. $52 = |G| = |C(x)| \cdot |Z(x)|$

orbit-stabilizer theorem for $G \curvearrowright G$ by conjugation.

$\Rightarrow \quad |Z(x)| = 52/4 = 13$

$x \in Z(x)$, $x \neq e$ (because $|C(x)| \neq 1$) $\Rightarrow |x| = 13$. $\square$.

2. $G_1/\ker\theta \xrightarrow{\sim} \theta(G_1) \leq G_2$

first isom. thm.

$\Rightarrow \quad |\theta(G_1)| \mid |G_2|$ (Lagrange's thm)

& $|\theta(G_1)| = |G_1/\ker\theta| = |G_1|/|\ker\theta| \Rightarrow |\theta(G_1)| \mid |G_1|$

$\Rightarrow \quad |\theta(G_1)| \mid \gcd(|G_1|, |G_2|)$. $\square$.

3. $G \curvearrowright X$, $|X| = 5 \quad \rightsquigarrow \quad \varphi: G \longrightarrow S_X \simeq S_5$

$g \longmapsto (x \mapsto g.x)$

$|\varphi(G)| \mid \gcd(|G|, |S_X|) = \gcd(90, \underset{5!}{\underset{\shortparallel}{120}}) = 30.$

$\underset{90}{\underset{\shortparallel}{|G|}} = |\ker\varphi| \cdot |\varphi(G)|$ (by first isom thm) $\Rightarrow |\ker\varphi| \neq 1$, $\ker\varphi \neq \{e\}$

Also $\ker\varphi \neq G$ because $G \curvearrowright X$ is non-trivial by assumption.

So $\ker\varphi \trianglelefteq G$, $\ker\varphi \neq \{e\}, G$ $\Rightarrow$ G not simple. $\square$.

4a. $H \leq Q_8$, $H \neq \{1\}$ $\Rightarrow$ $-1 \in H$: $Q_8 = \{\pm 1, \underbrace{\pm i, \pm j, \pm k}\}$

those elements satisfy $x^2 = -1$ □.

b. $Q_8 \hookrightarrow S_8$ by Cayley's thm (consider action of $G = Q_8$ on itself by left mult.)

Suppose $Q_8 \underset{\varphi}{\hookrightarrow} S_n$, $n < 8$, equivalently $\underset{G}{\overset{\shortparallel}{Q_8}} \curvearrowright \underset{X}{\overset{\shortparallel}{\{1,2,\dots,n\}}}$ faithful action.

OST $\Rightarrow$ $|G_x| > 1 \ \forall \ x \in X$ $(\because |G| = |O_x| \cdot |G_x|$, & $|O_x| \leq |X| < |G|)$

$\overset{(a)}{\Rightarrow}$ $-1 \in G_x \ \forall \ x \in X$

$\Rightarrow$ $\ker \varphi = \bigcap_{x \in X} G_x \ni -1$, $\ker \varphi \neq \{e\}$ ※ φ is injective.

5. a) $g \cdot x = x \Rightarrow g = e$ (cancellation law in group G)

Thus $\varphi(g)$ has cycle decomposition a product of n/k disjoint k-cycles, where $n = |G|$ & $k = |g|$.

So $\operatorname{sgn}(\varphi(g)) = (-1)^{(k-1) \cdot n/k} = -1 \iff k$ is even & n/k is odd □

b) $|G| = 2m$, m odd.

$\exists \ g \in G$. $|g| = 2$. (more generally, if G finite group & p is a prime dividing $|G|$, $\exists \ g \in G$. $|g| = p$ (e.g. follows from Sylow thm 1).)

Then $\operatorname{sgn}(\varphi(g)) = (-1)^m = -1$.

$$\Rightarrow \quad \begin{tikzcd} G \arrow[r, "\varphi"] \arrow[rr, bend right, "\theta"'] & S_G \arrow[r, "\operatorname{sgn}"] & \{\pm 1\} \end{tikzcd}, \quad \theta \text{ is surjective}$$

$\Rightarrow$ $\ker \theta \trianglelefteq G$, index 2. □.

6. p prime, G non-abelian, $|G| = p^3$.

Recall: $|G| = p^n \Rightarrow$ ~~$|Z(G)|$~~ $Z(G) \neq \{e\}$.

Also, by Q7, $|Z(G)| \neq p^2$, so $|Z(G)| = p$.

Suppose $x \in G \setminus Z(G)$ then $Z(G) \underset{\neq}{<} Z(x) \underset{\neq}{<} G$

$x \notin Z(G)$ $\qquad x \in Z(x)$ $\qquad x \notin Z(G)$

$\Rightarrow |Z(x)| = p^2 \Rightarrow |C(x)| = |G|/|Z(x)| = p$.

So class eq. is $|G| = p^3 = \sum_{C \text{ conj class}} |C| = \underbrace{(1 + \dots + 1)}_{p} + \underbrace{(p + \dots + p)}_{p^2-1}$ $\square$.

7. Let $\bar{a}$ be a generator of $G/Z(G)$ & lift to an element $a \in G$.

Given $g \in G$ its image $\bar{g} \in G/Z(G)$ is a power of $\bar{a}$, $\bar{g} = \bar{a}^k$, some $k \in \mathbb{Z}$

Then $g = a^k \cdot z$, some $z \in Z(G)$

Now $g_1 \cdot g_2 = a^{k_1} \cdot z_1 \cdot a^{k_2} \cdot z_2 \underset{z_1 \in Z(G)}{=} a^{k_1} \cdot a^{k_2} \cdot z_1 z_2 = a^{k_1+k_2} \cdot z_1 z_2$

$= a^{k_2+k_1} z_2 \cdot z_1 \underset{z_2 \in Z(G)}{=} a^{k_2} z_2 \cdot a^{k_1} z_1 = g_2 g_1$

$\Rightarrow G$ abelian. $\square$.

8. $|G/Z(G)| \mid 15 \Rightarrow G/Z(G)$ is cyclic $(\{e\}, \mathbb{Z}/3\mathbb{Z}, \mathbb{Z}/5\mathbb{Z}$, or $\mathbb{Z}/15\mathbb{Z})$

recall $|G'| = p \cdot q$, $p < q$, $q \not\equiv 1 \bmod p \Rightarrow G' \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/q\mathbb{Z} \cong \mathbb{Z}/pq\mathbb{Z}$.

Q7.
$\Rightarrow G$ abelian.

9. a. $Z(G) \neq \{e\} \Rightarrow |G/Z(G)| = 1, 3$ or $7 \Rightarrow G/Z(G)$ cyclic

$\Rightarrow G$ abelian ✱.

b. $21 = 1 + 3a + 7b$ some $a, b \in \mathbb{Z}_{\geq 0}$

(~~order~~ cardinalities of conjugacy classes divide $|G|$ & sum to $|G|$)

The only solution is $21 = 1 + 3 \cdot 2 + 7 \cdot 2$

i.e. Class Eq. is $21 = 1 + 3 + 3 + 7 + 7$.

10. Given $H \leq G$, $H \trianglelefteq N(H) \leq G$ & $N(H) \xrightarrow{\varphi} \mathrm{Aut}(H)$

$g \longmapsto (h \mapsto ghg^{-1})$

Our case $H = \langle x \rangle \leq G$.

$\exists\ g \in N(H)$ s.t. $gxg^{-1} = x^{-1}$

$\Rightarrow \varphi(g) \in \mathrm{Aut}\, H$ has order 2 (or $x = e$, or $\underset{x \neq e}{x^2 = e}$ ✱ $|G|$ odd)

$\Rightarrow g \in N(H) \leq G$ has even order ✱ $|G|$ odd.

□.

11. $|G| = 75 = 3 \cdot 5^2$, G non abelian.

$s := \#$ sylow 3-subgroups $\equiv 1 \bmod 3$, $s \mid 25 \Rightarrow s = 1$ or 25

$t := \#$ Sylow 5-subgroups $\equiv 1 \bmod 5$, $t \mid 3 \Rightarrow t = 1$.

If $s = t = 1$, have $H, K \trianglelefteq G$, $|H| = 3$, $|K| = 5^2$

$H \cap K = \{e\}$ (because $\gcd(|H|, |K|) = 1$, using Lagrange's thm)

$\Rightarrow H \times K \longrightarrow G$ injective map of sets $\Rightarrow HK = G$

$(h, k) \longmapsto hk$ (since $|H \times K| = |G|$)

Now, $H, K \trianglelefteq G$, $H \cap K = \{e\}$, $HK = G \Rightarrow H \times K \xrightarrow{\sim} G$ isom.

Also $H \simeq \mathbb{Z}/3\mathbb{Z}$, $K \simeq (\mathbb{Z}/5\mathbb{Z})^2$ or $\mathbb{Z}/25\mathbb{Z}$, in particular H & K abelian, so G abelian ✱

So, must have $s=25$

$\Rightarrow$ # elements of order 3 $= 25\cdot(3-1) = 50$.

12. G simple, $|G| = 168. = 2^3\cdot 3\cdot 7$

$s =$ # Sylow 7-subgroups $\equiv 1 \bmod 7$, $s \mid 24$ $\Rightarrow$ $s = 1$ or 8

$s \neq 1$ (otherwise Sylow 7-subgp is normal & G is not simple ※)

So $s=8$, # elements of order 7 $= 8\cdot(7-1) = 48$.

13. $|A_5| = 60 = 2^2\cdot 3\cdot 5$
$$\underset{\frac{1}{2}(5!)}{\overset{\parallel}{60}}$$

$H = \{e, (12)(34), (13)(24), (14)(23)\} \leq A_5$ is a Sylow 2-subgroup.

All Sylow p-subgroups are conjugate (Sylow Thm 2) $\Rightarrow$ $\# = \binom{5}{4} = 5$

(Alternatively, $|G| \overset{OST}{=} (\#\text{ conj subgps})\cdot |N(H)|$.

& $N(H) = S_4 \cap A_5 = A_4$, $|N(H)| = 12 \Rightarrow$ # conj subgps $= 5$)

14. $|G| = 44. = 2^2\cdot 11$

$s =$ # Sylow 2-subgps $\equiv 1 \bmod 2$, $s \mid 11 \Rightarrow s = 1$ or 11

$t =$ # Sylow 11-subgps $\equiv 1 \bmod 11$, $t \mid 4 \Rightarrow t = 1$.

G non abelian $\Rightarrow (s,t) \neq (1,1)$ (cf. Q 11), so $(s,t) = (11,1)$.

Let H, K be Sylow 2, 11 subgroups.

Then $\underset{(t=1)}{K \lhd G}$, $H \leq G$, $\underbrace{H\cap K = \{e\},\ HK = G}_{\text{cf. Q 11}}$ $\Rightarrow$ $G \simeq K \rtimes_\varphi H$,
some hom $\varphi: H \to \operatorname{Aut} K$.

Also $K \simeq \mathbb{Z}/11\mathbb{Z}$ & $H \simeq \mathbb{Z}/4\mathbb{Z}$ (b/c $\exists\, x\in G.\ |x| = 4$)

So $H \overset{\varphi}{\xrightarrow{\sim}} \operatorname{Aut} K$

$$\begin{array}{ccc} \| & & \| \\ \mathbb{Z}/4\mathbb{Z} & \xrightarrow{\psi} & (\mathbb{Z}/11\mathbb{Z})^\times \simeq \mathbb{Z}/10\mathbb{Z} \end{array}$$

$\Rightarrow \quad \psi(1) = -1 \in (\mathbb{Z}/11\mathbb{Z})^\times$ (ψ non trivial b/c G non abelian)

Thus $G \simeq \mathbb{Z}/11\mathbb{Z} \rtimes_\psi \mathbb{Z}/4\mathbb{Z}$ $\quad \psi(1) = (x \mapsto -x) \in \operatorname{Aut}(\mathbb{Z}/11\mathbb{Z})$

$\simeq \langle a, b \mid a^{11}, b^4, bab^{-1} = a^{-1} \rangle$

b) Each element of G can be written uniquely as $a^i b^j$, $0 \le i < 11$ $0 \le j < 4$.

$a \cdot a^i b^j = a^{i+1} b^j$, $\quad a^i b^j \cdot a = a^{i + (-1)^j} \cdot b^j$

$ba = a^{-1} b$

$b \cdot a^i b^j = a^{-i} b^{j+1}$, $\quad a^i b^j \cdot b = a^i b^{j+1}$

Thus $a^i b^j \in Z(G) \iff j$ even, $2i \equiv 0 \mod 11$

$\iff j = 0, 2, \ i = 0.$

So. $Z(G) = \{e, b^2\} = \langle b^2 \rangle$

c) $G/Z(G) \simeq \langle a, b \mid a^{11}, b^2, bab^{-1} = a^{-1} \rangle \simeq D_{11}$ $\square$.

15. $|G| = 28$, G non abelian $\nexists x \in G.\ |x| = 4.$
$= 2^2 \cdot 7$

$s :=$ # Sylow 2-subgps, $\quad s \equiv 1 \mod 2$, $s \mid 7 \Rightarrow s = 1$ or 7
$t :=$ # Sylow 7-subgps, $\quad t \equiv 1 \mod 7$, $t \mid 4 \Rightarrow t = 1$

G non-abelian $\Rightarrow (s,t) = (7,1)$ (cf. Q11)

$\nexists x \in G.\ |x| = 4 \Rightarrow$ Sylow 2-subgp $H \simeq (\mathbb{Z}/2\mathbb{Z})^2$, also Sylow 7-subgp $K \simeq \mathbb{Z}/7\mathbb{Z}$

$K \trianglelefteq G,\ H \leq G,\ H \cap K = \{e\},\ HK = G \implies G \simeq K \rtimes_\varphi H$

$\varphi : H \to \operatorname{Aut} K$

$$\begin{array}{ccc} H & \xrightarrow{\varphi} & \operatorname{Aut}K \\ \wr\wr & & \wr\wr \\ (\mathbb{Z}/2\mathbb{Z})^2 & \xrightarrow[\psi]{} & \operatorname{Aut}(\mathbb{Z}/7\mathbb{Z}) \simeq (\mathbb{Z}/7\mathbb{Z})^\times \simeq \mathbb{Z}/6\mathbb{Z} \end{array}$$

The hom $(\mathbb{Z}/2\mathbb{Z})^2 \to \mathbb{Z}/6\mathbb{Z}$ is necessarily given by composition $(\mathbb{Z}/2\mathbb{Z})^2 \to \mathbb{Z}/2\mathbb{Z} \xrightarrow{\times 3} \mathbb{Z}/6\mathbb{Z}$

& $(\mathbb{Z}/2\mathbb{Z})^2 \to (\mathbb{Z}/2\mathbb{Z})$ is given by $(x,y) \mapsto x$ after change of basis in domain & target, so $\exists!$ isom. type of G, given by

again note ψ nontrivial since G non abelian

$$\cancel{G \simeq (\mathbb{Z}/2\mathbb{Z})^2 \rtimes_\psi \mathbb{Z}/6\mathbb{Z} \simeq \langle a,b,c \mid a^2, b^2, ab=ba, cac^{-1}=a^{-1}, cbc^{-1}=b}$$

$$G \simeq (\mathbb{Z}/7\mathbb{Z}) \rtimes_\psi (\mathbb{Z}/2\mathbb{Z})^2 \simeq \langle a,b,c \mid a^7, b^2, c^2, bc=cb, bab^{-1}=a^{-1}, cac^{-1}=a\rangle$$

$$\simeq \langle a,b \mid a^7, b^2, bab^{-1}=a^{-1}\rangle \times \mathbb{Z}/2\mathbb{Z}$$

$$\simeq D_7 \times \mathbb{Z}/2\mathbb{Z}. \quad \square$$

16. $|G| = 18 = 2 \cdot 3^2$

$s = \#$ Sylow 2-subgroups $\equiv 1 \bmod 2$, $s \mid 9 \implies s = 1, 3$ or 9

$t = \#$ Sylow 3-subgroups $\equiv 1 \bmod 3$, $t \mid 2 \implies t = 1$.

H Sylow 2-subgp, K Sylow 3 subgp, then $K \simeq (\mathbb{Z}/3\mathbb{Z})^2$ $(\nexists\, x \in G,\ |x| = 9)$

$H \simeq \mathbb{Z}/2\mathbb{Z}$.

$K \trianglelefteq G,\ H \leq G,\ H \cap K = \{e\},\ HK = G \implies G \simeq K \rtimes_\varphi H$

$\simeq (\mathbb{Z}/3\mathbb{Z})^2 \rtimes_\psi \mathbb{Z}/2\mathbb{Z}$

$\psi : \mathbb{Z}/2\mathbb{Z} \to \operatorname{Aut}(\mathbb{Z}/3\mathbb{Z})^2 \simeq GL_2(\mathbb{Z}/3\mathbb{Z})$

$\psi(1) = A$, $A^2 = I$, $\underbrace{(A-I)(A+I)}_{\text{distinct linear factors}} = 0 \implies A \underset{\text{similar}}{\sim} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ (NB. $A \neq I$, o/w G abelian)

$\Rightarrow \quad G \simeq \langle a,b,c \mid a^3, b^3, c^2, ab=ba,\ cac^{-1}=a,\ cbc^{-1}=b^{-1} \rangle$

$\simeq \langle b,c \mid b^3, c^2,\ cbc^{-1}=b^{-1} \rangle \times \mathbb{Z}/3\mathbb{Z}$

$\simeq D_3 \times \mathbb{Z}/3\mathbb{Z}$

OR $\quad G \simeq \langle a,b,c \mid a^3, b^3, c^2, ab=ba,\ cac^{-1}=a^{-1},\ cbc^{-1}=b^{-1} \rangle$. $\square$

$|G| = pq^2$

17. $s = \#$ Sylow p-subgroups

$s \equiv 1 \bmod p$ & $s \mid q^2$

$t = \#$ Sylow q-subgroups

$t \equiv 1 \bmod q$ & $t \mid p$

If $s \neq 1$ & $t \neq 1$ then $t = p$, $p \equiv 1 \bmod q \Rightarrow p > q$

$\Rightarrow s = q^2$, $q^2 \equiv 1 \bmod p \quad (s \equiv 1 \bmod p \Rightarrow s > q)$

~~Now $p \mid (q^2-1) = (q+1)(q-1) \Rightarrow p \mid q+1$ (or $p \mid q-1$ ⨳ $p>q$)~~

~~& $q \mid (p-1) \Rightarrow p = q+1 \quad (p>q)$~~

~~So $q=2$, $p=3$, $|G|=12$.~~

Now $\#$ elements of order p = $q^2 \cdot (p-1)$.

Remaining q^2 elements form unique Sylow q-subgp, i.e $t=1$. ⨳ $\square$.

18. $G = \left(\mathbb{Z}/q\mathbb{Z}\right)^2 \rtimes_\psi \mathbb{Z}/p\mathbb{Z}$

where $\psi(1) \in \mathrm{Aut}\left((\mathbb{Z}/q\mathbb{Z})^2\right) \cong GL_2(\mathbb{Z}/q\mathbb{Z})$ is an element of order p.

N.B. $p \mid |GL_2(\mathbb{Z}/q\mathbb{Z})| = (q^2-1)(q^2-q)$ by assumption $q^2 \equiv 1 \bmod p$

$\Rightarrow \exists$ element of order p in $GL_2(\mathbb{Z}/q\mathbb{Z})$. $\square$.

19. a $|G| = 40. = 2^3 \cdot 5$

$s =$ # Sylow 2 subgps $\equiv 1 \mod 2$, $s | 5 \Rightarrow s = 1, 5$

$t =$ # Sylow 5-subgps $\equiv 1 \mod 5$, $t | 8 \Rightarrow t = 1$.

$H :=$ Sylow 5-subgp, $H \cong \mathbb{Z}/5\mathbb{Z}$, $H \trianglelefteq G$,

$|G/H| = 2^3 \Rightarrow G/H$ solvable. ($|G| = p^n \Rightarrow$ solvable)

G/H & H solvable $\Rightarrow G$ solvable

b. $|G| = 48 = 2^4 \cdot 3$

$s =$ # Sylow 2 subgps $\equiv 1 \mod 2$, $s | 3 \Rightarrow s = 1$ or 3.

if $s = 1$, $H \trianglelefteq G$, $|H| = 2^4 \Rightarrow H$ solvable
& $|G/H| = 3$, $G/H \cong \mathbb{Z}/3\mathbb{Z}$
$\Rightarrow G$ solvable.

if $s = 3$. $G \curvearrowright X = \{$ Sylow 2 subgps $\}$ transitive (by Sylow Thm 2)

$\Rightarrow G \xrightarrow{\varphi} S_3$ non trivial hom, $|\varphi(G)| = 3$ or 6

$H := \ker\varphi \trianglelefteq G$, $|\ker\varphi| = 2^4$ or $2^3 \Rightarrow$ solvable.

$G/H \underset{\text{F.I.T.}}{\hookrightarrow} S_3$, S_3 solvable $\Rightarrow G/H$ solvable

$\Rightarrow G$ solvable. $\square$.

20. G finite group. p smallest prime dividing $|G|$. $H \leq G$, $[G:H] = p$.

$G \curvearrowright G/H$ $\quad g \cdot (aH) := (g \cdot a)H$

$\leadsto \varphi : G \to S_p$

$\gcd(|G|, |S_p|) = \gcd(|G|, p!) = p$

$\Rightarrow |\varphi(G)| \mid p. \Rightarrow |\ker\varphi| \geq |G|/p = |H|$

Now $\ker\varphi \leq G_H = H$, $\Rightarrow \ker\varphi = H$, H normal. $\square$.