Thursday 10/3/19.

MATH 611 HW2 SOLUTIONS.

3. $G \curvearrowright X$, $p \nmid |X|$, $|G| = p^n$

X is partitioned into orbits, $|X| = |O_1| + |O_2| + \dots + |O_k|$.

For each orbit O, $|O| \mid |G|$, $\Rightarrow$ $|O| = p^\alpha$, $\alpha \leq n$.
(by orbit-stabilizer thm)

If $G \curvearrowright X$ has no fixed points, $|O| \neq 1$ $\forall$ orbits O,

$\Rightarrow$ $p \mid |O|$ $\forall$ O $\Rightarrow$ $p \mid |X|$ ※ □

4. G group, $a \in G$.

$Z(a) := \{ g \in G \mid ga = ag \} \leq G$.

For $G \curvearrowright G$ by conjugation,

$Z(a) = G_a$, stabilizer of a

$C(a) = O_a$, orbit of a.

In particular, $|G| = |C(a)| \cdot |Z(a)|$ by OST.

a) $a = (123) \in S_5$

$g \in Z(a) \Leftrightarrow gag^{-1} = a$, i.e. $(g(1)g(2)g(3)) = (123)$, $\Leftrightarrow$ $g \in \langle (123), (45) \rangle$.

i.e. $Z(a) = \langle (123), (45) \rangle$, $|Z(a)| = 6$, $|C(a)| = |S_5| / 6$

$\cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ $= 5!/6$

$= 20$.

(Check: $|C(a)| = \#\ 3\text{ cycles} = \dfrac{5 \cdot 4 \cdot 3}{3} = 20$ ✓)

b) $a = (123)(456) \in S_7$

$Z(a) = \langle (123), (456), (14)(25)(36) \rangle$

$\cong (\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}$, $\varphi: \mathbb{Z}/2\mathbb{Z} \to \mathrm{Aut}((\mathbb{Z}/3\mathbb{Z})^2)$ $\varphi(1) = ((x,y) \mapsto (y,x))$

$$|Z(a)| = 3\cdot 3\cdot 2 = 18 \Rightarrow |C(a)| = 7!/_{18} = \frac{7\cdot \cancel{6}^{2}\cdot 5\cdot 4\cdot \cancel{3}\cdot \cancel{2}}{\cancel{3}\cdot \cancel{3}\cdot \cancel{2}}$$

$$= 280.$$

$$\left(\text{Check: } |C(a)| = \#\{(abc)(def) \in S_7\} = \frac{\left(\frac{7\cdot 6\cdot 5}{3}\right)\cdot\left(\frac{4\cdot 3\cdot 2}{3}\right)}{2} = \frac{70\cdot 8}{2} = 280 \checkmark. \right)$$

c)
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\cdot\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\cdot\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 2a & a+2b \\ 2c & c+2d \end{pmatrix} = \begin{pmatrix} 2a+c & 2b+d \\ 2c & 2d \end{pmatrix}$$

$$\Leftrightarrow c=0,\ a=d$$

$$Z\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \;\middle|\; 0\neq a\in \mathbb{Z}/_{5\mathbb{Z}},\ b\in \mathbb{Z}/_{5\mathbb{Z}} \right\}$$

$$\left|Z\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\right| = 4\cdot 5 = 20 \qquad \left|C\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\right| = \frac{|GL_2(\mathbb{Z}/_{5\mathbb{Z}})|}{20} = \frac{(5^2-1)(5^2-5)}{20}$$

$$= 24.$$

d.)
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} b & -a \\ d & -c \end{pmatrix} = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$$

$$\Leftrightarrow a=d,\ b=-c$$

i.e. $$Z\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \left\{ A=\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \;\middle|\; \overset{\det A}{\overset{\shortparallel}{a^2+b^2}} \cancel{\neq 0} = 1,\ a,b\in \mathbb{Z}/_{3\mathbb{Z}} \right\}$$

$$\Rightarrow (a,b) = (\pm 1, 0),\ (0,\pm 1)$$

$$\therefore \left|Z\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right| = 4, \quad \left|C\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right| = \frac{|SL_2(\mathbb{Z}/_{3\mathbb{Z}})|}{4} = \frac{(3^2-1)(3^2-3)}{(3-1)} \Big/ 4 = 6.$$

5. $|G| = 21.$ $\left.\begin{array}{l} \\ |C(x)| = 3 \end{array}\right\} \Rightarrow \quad |Z(x)| = 7$

$x \in Z(x)$, $x \neq e$ (otherwise $|C(x)| = |\{x\}| = 1$), so $|x| = 7$.

7. S_4 . Conjugacy classes:

$[e]$ $\quad 1$

$[(12)]$ $\quad 4\cdot3/2 = 6$

$[(123)]$ $\quad 4\cdot3\cdot2/3 = 8$

$[(1234)]$ $\quad 4\cdot3\cdot2\cdot1/4 = 6$

$[(12)(34)]$ $\quad (4\cdot3/2)\cdot(2\cdot1/2)/2 = 3$

$\therefore$ Class eq. $\quad 24 = 1 + 6 + 8 + 6 + 3$

If $H \trianglelefteq S_4$, H is a union of conjugacy classes, & $|H| \mid |S_4| = 24$, $e \in H$.

Possibilities: $\quad |H| = 1,\ 1+3,\ 1+3+8,\ 1+3+6+6+8$

$\rightsquigarrow \quad H = \{e\},\ \{e, (12)(34), (13)(24), (14)(23)\},\ A_4,\ S_4.$ $\square$

$\underset{(\mathbb{Z}/2\mathbb{Z})^2}{\overset{}{\cong}}$

9. $G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b, c \in \mathbb{Z}/p\mathbb{Z} \right\} \leq GL_3(\mathbb{Z}/p\mathbb{Z})$

$|G| = p^3$.

a). $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a+a' & b+b'+ac' \\ 0 & 1 & c+c' \\ 0 & 0 & 1 \end{pmatrix}$

So $\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ & $\begin{pmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{pmatrix}$ commute iff $ac' = a'c$.

Thus $Z(G) = \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; b \in \mathbb{Z}/p\mathbb{Z} \right\}$.

b. $G/Z(G) \xrightarrow{\sim} (\mathbb{Z}/p\mathbb{Z})^2$

$$\left[\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}\right] \longmapsto (a,c)$$

$\left(G \twoheadrightarrow (\mathbb{Z}/p\mathbb{Z})^2, \quad \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \longmapsto (a,c) \quad \text{surj. hom, } \ker = Z(G) \overset{FIT}{\Rightarrow} \square \right)$

10. Use the class equation.

$\{e\}$ is a conjugacy class.

So 1 conj class $\Rightarrow$ $\underline{G = \{e\}}$

2 conj classes: $\underbrace{|G| = n = 1 + (n-1)}_{CE}$ $\Rightarrow n-1 \mid n$

$\Rightarrow n = 2$ $(\gcd(n, n-1) = 1!)$

$\underline{G \simeq \mathbb{Z}/2\mathbb{Z}}$

3 conj. classes: $|G| = \overbrace{n = 1 + a + b}^{CE}$

WLOG $1 \leq a \leq b$.

$a, b \mid n \Rightarrow b \mid 1+a, \quad a \mid 1+b.$

$\Rightarrow a \leq b \leq 1+a$, $b = a$ OR $a+1$.

If $b=a$, $a \mid 1+a$, $a \mid 1$, $a = 1$. $|G| = 1+1+1 = 3$, $\underline{G \simeq \mathbb{Z}/3\mathbb{Z}}$.

If $b = a+1$, $a \mid 2+a$, $a \mid 2$, $a = 1$ or 2. $\underline{a=1}$: $|G| = 1+1+2 = 4$, $\#$ $|G|=4 \Rightarrow$ abelian.

$\underline{a=2}$

$|G| = 1+2+3 = 6$, $\underline{G \simeq S_3}$

$\square$.

12. a.

Counter-example:

$$K = \langle (12) \rangle \trianglelefteq H = \{e, (12)(34), (13)(24), (14)(23)\} \trianglelefteq G = S_4$$

but $K \ntrianglelefteq S_4$.

b. True: Recall – $H \trianglelefteq G \iff gHg^{-1} = H \quad \forall g \in G$

$\iff gHg^{-1} \subset H \quad \forall g \in G.$

Suppose $H \trianglelefteq G$ & $K \leq G$.

Let $k \in K$, then $k(H \cap K)k^{-1} \subset kHk^{-1} \subset H \quad \because H \trianglelefteq G$

& $k(H \cap K)k^{-1} \subset K \quad \because K \leq G$

So $k(H \cap K)k^{-1} \subset H \cap K$, $\forall k \in K$, $\therefore H \cap K \trianglelefteq K$ as claimed. $\square$

(Alternative: $H \trianglelefteq G \iff \exists \; \varphi : G \to G'$ s.t. $H = \ker \varphi$.

Now $\ker(\varphi|_K) = \ker \varphi \cap K = H \cap K$

$\Rightarrow H \cap K \trianglelefteq K.$ $\square$)