# Math 611 Midterm review problems

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(1) Let $p$ be a prime and $G$ a non-abelian group of order $p^3$. Determine the class equation of $G$.

(2) Let $G$ be a non-abelian group of order 21.

  (a) Prove that the center of $G$ is trivial.

  (b) Determine the class equation of $G$.

(3) (a) Show that any non-trivial subgroup of $Q_8$ contains the element $-1$.

  (b) Show that $Q_8$ is isomorphic to a subgroup of $S_8$, but is not isomorphic to a subgroup of $S_n$ for any $n < 8$.

(4) Let $G$ be a finite group of odd order and $x \in G$ an element. Show that if $x$ and $x^{-1}$ are conjugate then $x = e$.

(5) Let $G$ be a group of order 60 such that the order of the center of $G$ is divisible by 4. Prove that $G$ is abelian.

(6) Let $G$ be a simple group of order 168. Determine the number of elements of order 7 in $G$.

(7) Let $G$ be a group of order 20. Suppose $G$ contains an element of order 4 and has trivial center. Describe $G$ in terms of generators and relations.

(8) Let $G$ be a finite group, $N$ a normal subgroup of $G$, and $p$ a prime such that $p$ divides the order of $G/N$. Show that the number of Sylow $p$-subgroups of $G/N$ is less than or equal to the number of Sylow $p$-subgroups of $G$.

(9) Let $G$ be a group of order $p^2q$ where $p$ and $q$ are distinct primes. Show that one of the Sylow subgroups of $G$ is normal.

(10) Determine the number of Sylow 2-subgroups in the alternating group $A_5$.

(11) Show that a group of order (a) 40 (b) 48 is solvable.

[Note: Actually it is a theorem of Burnside that any group of order $p^a q^b$ is solvable. But please prove these special cases without appealing to Burnside's theorem.]

(12) Show that there is no simple group of order 120.

(13) Let $G$ be a finite group and $p$ a prime dividing $|G|$. Suppose $H$ is a subgroup of $G$ of index $p$.

(a) What are the possibilities for the number of conjugate subgroups of $H$?

(b) Suppose in addition that $p$ is the smallest prime dividing $|G|$. Prove that $H$ is normal.

(14) Let $G = \langle x, y, z \mid yz^2xy \rangle$ be the group generated by $x, y, z$ subject to the relation $yz^2xy = e$. Prove that $G$ is isomorphic to the free group generated by two elements.

(15) In each of the following cases, identify the group described by generators and relations with a standard group.

(a) $\langle a, b \mid a^5 = b^2 = (ab)^2 = e \rangle$.

(b) $\langle a, b \mid a^4 = e,\ a^2 = b^2,\ ba = a^{-1}b \rangle$.

[Hint: First guess the standard group $G$ and a set of two generators $A, B \in G$ satisfying the given relations. Let $\theta : F/N \to G$ be the surjective homomorphism from the abstractly defined group to $G$ determined by $\theta(a) = A$, $\theta(b) = B$ (using the universal property of the free group). Show that $\theta$ is injective and so an isomorphism.]

(16) Let $G = \langle x, y \mid x^2, y^2 \rangle$ be the group generated by $x$ and $y$ subject to the relations $x^2 = e$ and $y^2 = e$. Describe an isomorphism $\theta$ from $G$ to a semi-direct product of two abelian groups.