Math 611 Homework 8

Paul Hacking

December 4, 2015

All rings are assumed to be commutative with 1.

(1) Let R be a ring and M an R-module. We say M is *cyclic* if it is generated as an R-module by a single element $m \in M$.

(a) Show that if M is cyclic then M is isomorphic to R/I for some ideal $I \subset R$.

(b) Let M be the $\mathbb{R}[x]$-module determined by the $\mathbb{R}$-vector space $V = \mathbb{R}^2$ and the linear transformation T given by the matrix $A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$. Is M a cyclic module?

(2) For each of the following abelian groups, determine an isomorphic direct product of cyclic groups.

(a) $\mathbb{Z}^3/A\mathbb{Z}^2$, where $A = \begin{pmatrix} 3 & 2 \\ 2 & 0 \\ 8 & 4 \end{pmatrix}$

(b) $\mathbb{Z}^3/A\mathbb{Z}^3$, where $A = \begin{pmatrix} -4 & -6 & 7 \\ 2 & 2 & 4 \\ 6 & 6 & 15 \end{pmatrix}$.

(3) Let $R = \mathbb{C}[x]$ and let M be the R-module $M = R^2/AR^2$, where

$$A = \begin{pmatrix} x+1 & 2x+3 \\ x^3+x+2 & 2x^3+x+6 \end{pmatrix}.$$

Determine a direct sum of cyclic R-modules which is isomorphic to M. Use the Chinese remainder theorem to further decompose M if possible.

(4) Let $R = \mathbb{Z}[i]$ and let M be the R-module $M = R^2/AR^2$ where

$$A = \begin{pmatrix} 1+i & 3 \\ 2-i & 5i \end{pmatrix}.$$

Determine a direct sum of cyclic R-modules which is isomorphic to M. Use the Chinese remainder theorem to further decompose M if possible.

(5) Let $M = \mathbb{Z}^3$ and let $N \subset M$ be the subgroup generated by the elements

$$m_1 = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, m_2 = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}, m_3 = \begin{pmatrix} 5 \\ 7 \\ 1 \end{pmatrix}.$$

(a) Determine the isomorphism type of the abelian groups N and M/N. (Identify each group with a direct sum of copies of $\mathbb{Z}$ and $\mathbb{Z}/p^\alpha\mathbb{Z}$ for p prime.)

(b) Does there exist a submodule $L \subset M$ such that $M = L \oplus N$?

(6) (a) Let R be a ring and $I \subset R$ an ideal. Describe a bijective correspondence between R/I-modules and R-modules M such that $x \cdot m = 0$ for all $x \in I$ and $m \in M$.

(b) Let F be a field. Describe the classification of finitely generated modules over the ring $F[x]/(x^2)$.

(7) Let R be a PID and L, M, N finitely generated R-modules. Show that if $L \oplus N \simeq M \oplus N$ then $L \simeq M$.

(8) Classify matrices $A \in \mathrm{GL}_4(\mathbb{Q})$ of orders 4 and 5 up to conjugacy $A \rightsquigarrow P^{-1}AP$.

(9) Suppose $A \in \mathrm{GL}_n(\mathbb{Q})$ satisfies $A^8 = 9I$. Show that n is divisible by 4 and give an explicit example of such a matrix A for $n = 4$.

(10) Let p be a prime. Determine the number of conjugacy classes in $\mathrm{GL}_2(\mathbb{Z}/p\mathbb{Z})$.

(11) Let $J(m,\lambda)$ denote the $m \times m$ matrix

$$\begin{pmatrix} \lambda & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda & 0 \\ 0 & 0 & 0 & \cdots & 1 & \lambda \end{pmatrix}$$

We call $J(m,\lambda)$ the *Jordan block* of size m with eigenvalue λ. [Note: In DF they use the transpose of the above matrix. This corresponds to a change of basis given by reversing the order of the basis.]

Show that λ is the unique eigenvalue of $J(m,\lambda)$ and

$$\dim\ker(J(m,\lambda)-\lambda I)^k = \min(k,m).$$

(12) Let $F = \mathbb{C}$ (or any algebraically closed field) and $A \in F^{n\times n}$ a square matrix with entries in F. What is the *Jordan normal form* of A? Suppose that the eigenvalues of A are λ_i, $i = 1,\ldots,r$, and the sizes of the Jordan blocks with eigenvalue λ_i are $m_{i1} \le m_{i2} \le \ldots \le m_{is_i}$.

(a) What is the characteristic polynomial of A? What is the minimal polynomial of A?

(b) Let $d_{ik} = \dim\ker(A-\lambda_i I)^k$. Using Q11 or otherwise, describe an algorithm to determine the block sizes m_{ij} in terms of the dimensions d_{ik}.

(c) Determine the Jordan normal form of the matrix

$$A = \begin{pmatrix} 4 & 1 & 1 \\ -10 & -3 & -5 \\ 6 & 3 & 5 \end{pmatrix}$$

(13) Let R be a UFD and $f,g \in R$ nonzero elements such that $\gcd(f,g) = 1$. Let $I = (f,g) \subset R$ denote the ideal generated by f and g. [Warning: We do *not* assume that R is a PID so $I \neq R$ in general.] Prove that the sequence of R-modules

$$0 \to R \overset{\alpha}{\to} R^2 \overset{\beta}{\to} I \to 0$$

given by

$$\alpha(a) = (ag, -af)$$

and

$$\beta(a, b) = af + bg$$

is exact.

(14) Let $\delta = \sqrt{-5}$, $R = \mathbb{Z}[\delta]$ and $M = (2, 1+\delta) \subset R$. Determine a presentation for the R-module M, that is, a matrix $A \in R^{m\times n}$ for some m, n such that $M \simeq R^m/A \cdot R^n$. [Warning: R is *not* a UFD].

(15) Let F be a field and $R = F[x, y, z]$. Let $m = (x, y, z) \subset R$ be the maximal ideal generated by x, y, and z. Consider the sequence of R-modules

$$R^3 \xrightarrow{\beta} R^3 \xrightarrow{\gamma} m \to 0$$

where

$$\gamma(a, b, c) = ax + by + cz$$

and the homomorphism β is given by the matrix

$$\begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}.$$

(a) Show that the sequence is exact.

(b) Determine the kernel of β and use your result to describe an exact sequence

$$0 \to R \xrightarrow{\alpha} R^3 \xrightarrow{\beta} R^3 \xrightarrow{\gamma} m \to 0.$$

Hints:

(1) (b) What is the rational canonical form of A?

(7) What is the uniqueness statement in the structure theorem of finitely generated modules over a PID?

(8) Two matrices are conjugate iff they have the same rational canonical form. If $A^k = I$ what can you say about the minimal polynomial of A?

(9) What are the possibilities for the minimal polynomial of A?

(10) Use rational canonical form of matrices over $F = \mathbb{Z}/p\mathbb{Z}$.