Math 611 Homework 6

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All rings are assumed to be commutative with 1.

(1) Let R be a integral domain. We say an element $0 \neq a \in R$ is *irreducible* if a is not a unit and there does not exist a factorization $a = bc$ with $b, c \in R$ such that b and c are not units.

(a) What are the irreducible elements in (i) $\mathbb{C}[x]$? (ii) $\mathbb{R}[x]$?

(b) Show that if R is a principal ideal domain and R is not a field then the maximal ideals in R are the principal ideals (a) for $a \in R$ an irreducible element. (Note: this follows directly from definitions without using the statement "PID $\Rightarrow$ UFD" proved in class.)

(2) We say a ring R is *Noetherian* if there does not exist an infinite ascending chain

$$I_1 \subsetneq I_2 \subsetneq I_3 \subsetneq \cdots$$

of ideals of R. Equivalently, every ideal I of R is finitely generated, that is, there exist $a_1, a_2, \ldots, a_n \in R$ such that

$$I = (a_1, a_2, \ldots, a_n) := \{r_1 a_1 + r_2 a_2 + \cdots + r_n a_n \mid r_1, \ldots, r_n \in R\}.$$

(a) Show that $\mathbb{Z}$ is Noetherian, and any field F is Noetherian.

(b) The Hilbert basis theorem (DF p. 316) states that if R is Noetherian then $R[x]$ is Noetherian. Deduce that the polynomial rings $\mathbb{Z}[x_1, \ldots, x_n]$ and $F[x_1, \ldots, x_n]$ are Noetherian (where $n \in \mathbb{N}$ and F is a field).

(c) Show that if R is a Noetherian ring and $I \subset R$ is an ideal then R/I is Noetherian.

(d) Show that if R is a Noetherian ring, S is any ring, and $\varphi \colon R \to S$ is a ring homomorphism, then the image $\varphi(R)$ of φ (a subring of S) is a Noetherian ring.

(e) Let R be a ring. Suppose there exist a *coefficient ring* $A = \mathbb{Z}$ or F, a field, and elements $\alpha_1, \ldots, \alpha_n \in R$ such that every element of R may be expressed as a finite sum

$$\sum a_{i_1,\ldots,i_n} \alpha_1^{i_1} \cdots \alpha_n^{i_n}$$

with coefficients $a_{i_1,\ldots,i_n} \in A$. (Here in the case $A = F$ we assume that F is a subring of R.) Show that R is Noetherian.

(3) In this question we will show by example that a subring of a Noetherian ring is not necessarily Noetherian. Let F be a field and $S = F[x, y]$. Then S is Noetherian by the Hilbert basis theorem. Define

$$R := \left\{ \sum a_{ij} x^i y^j \in S \;\middle|\; a_{0j} = 0 \text{ for all } j > 0 \right\}.$$

(a) Show that R is a subring of S.

(b) Show that R is *not* Noetherian.

(4) We say a ring R is *Artinian* if there does not exist an infinite descending chain

$$I_1 \supsetneq I_2 \supsetneq I_3 \supsetneq \cdots$$

of ideals of R. While this condition appears to be similar to the Noetherian condition, it is actually *much* more restrictive!

(a) Show that $\mathbb{Z}$ is not Artinian.

(b) Let R be a ring, $R \neq \{0\}$. Show that $R[x]$ is not Artinian.

(c) Suppose R is a ring such that R is a finite set. Show that R is Artinian.

(d) Suppose F is a field and R is a ring containing F as a subring. Then R has the structure of an F-vector space (with the scalar multiplication $\lambda \cdot v$ for $\lambda \in F$ and $v \in R$ being given by multiplication in the ring R). Suppose that R is finite dimensional as an F-vector space. Show that R is Artinian.

(e) Let n be a positive integer. Show that $\mathbb{Z}/n\mathbb{Z}$ is Artinian

(f) Let F be a field and $0 \neq f \in F[x]$ a nonzero polynomial. Show that the quotient ring $F[x]/(f)$ is Artinian.

(5) Let R be a ring and $a \in R$ an element of R. Consider the quotient ring

$$R_a := R[x]/(ax-1).$$

This is the ring obtained from R by formally inverting the element $a \in R$. Let $\varphi\colon R \to R_a$ be the composition of the inclusion $R \subset R[x]$ and the quotient map $R[x] \to R_a$.

(a) Show that if R is an integral domain, $F = \operatorname{ff} R$ is its fraction field, and $a \neq 0$, then R_a is isomorphic to the subring S of F given by

$$S = \{b/a^n \mid b \in R, n \in \mathbb{Z}_{\geq 0}\} \subset F.$$

(b) In general show that the kernel of φ is given by

$$\ker(\varphi) = (ax-1) \cap R = \{b \in R \mid a^n b = 0 \text{ for some } n \in \mathbb{N}\}.$$

(c) Let $R = \mathbb{C}[s,t]/(st)$ and $a = s \in R$. (More carefully, I mean that a is the image of s in R under the quotient map $\mathbb{C}[s,t] \to R$). Describe R_a as a subring of a standard ring.

(6) Let

$$R = \mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\} \subset \mathbb{C},$$

a subring of $\mathbb{C}$. Prove that R is a UFD.

(7) Let $d \in \mathbb{Z}$ be an integer. Prove that

$$R := \{a + b(1+\sqrt{d})/2 \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$$

is a subring of $\mathbb{C}$ iff $d \equiv 1 \bmod 4$.

(8) Let $\omega = \frac{1}{2}(-1+\sqrt{-3})$, a primitive cube root of unity, and

$$R = \mathbb{Z}[\omega] = \{a + b\omega \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}.$$

(This is a subring of $\mathbb{C}$ by Q7.) Prove that R is a UFD.

(9) Let R be a UFD and $F = \mathrm{ff}(R)$ its field of fractions. Suppose $f \in R[x]$ is a monic polynomial (a polynomial with leading coefficient equal to 1). Suppose $\alpha \in F$ satisfies $f(\alpha) = 0$. Prove that $\alpha \in R$.

(10) Let

$$R = \mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} \mid a, b \in \mathbb{Z}\} \subset \mathbb{R},$$

a subring of $\mathbb{R}$. Show that R is not a UFD.

(11) Let

$$R = \mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\} \subset \mathbb{R},$$

a subring of $\mathbb{R}$. Define

$$\theta\colon R \to R, \quad \theta(a + b\sqrt{2}) = a - b\sqrt{2}$$

and

$$\sigma\colon R \to \mathbb{Z}_{\geq 0}, \quad \sigma(\alpha) = |\alpha \cdot \theta(\alpha)|,$$

explicitly

$$\sigma(a + b\sqrt{2}) = |a^2 - 2b^2|.$$

(a) Show that θ is a ring homomorphism. Deduce that $\sigma(\alpha\beta) = \sigma(\alpha)\sigma(\beta)$ for all $\alpha, \beta \in R$.

(b) Show that $\sigma(\alpha) \neq 0$ for $\alpha \neq 0$.

(c) Show that $\alpha \in R$ is a unit iff $\sigma(\alpha) = 1$.

(d) Find a unit $\alpha \in R$, and use it to prove that there are infinitely many units in R.

(e) (Optional) Show that R is a UFD.

(12) Let

$$R = \mathbb{C}[x, y]/(y^2 - x^3).$$

Prove carefully that R is not a UFD.

Hints:

(2) (b) Recall $R[x,y] = (R[x])[y]$. (d) Use the first isomorphism theorem. (e) Use (b) and (d).

(3) (b) For $n \in \mathbb{Z}_{\geq 0}$ define the ideal $I_n = (x, xy, xy^2, \ldots, xy^n) \subset R$. Prove that $I_n \subsetneq I_{n+1}$ for all $n \geq 0$.

(4) (d) If $I \subset R$ is an ideal then in particular I is a subspace of the F-vector space R (why?). (f) What is a basis for $F[x]/(f)$ as an F-vector space?

(5) (a) Use the first isomorphism theorem. (b) Compute the intersection $(ax-1) \cap R$ in $R[x]$ explicitly. (c) $R_a = \mathbb{C}[s,t,x]/(st, sx-1)$. What can you say about the element $t \in R_a$?

(6) Adapt the geometric proof that $\mathbb{Z}[i]$ is a UFD given in class.

(8) Again there is a geometric proof similar to the case of $\mathbb{Z}[i]$.

(9) You may be familiar with this result in the case $R = \mathbb{Z}$. The proof in the general case is the same.

(10) This can be proved using the result of Q9.

(11) (d) The units form a group under multiplication. (e) Show that R is an ED for the size function σ. Given $\alpha, \beta \in R$, $\beta \neq 0$, we have $\alpha/\beta = x + y\sqrt{2}$ with $x, y \in \mathbb{Q}$. So there exists $q \in R$ such that $\alpha/\beta - q = u + v\sqrt{2}$ with $|u|, |v| \leq 1/2$.

(12) The ring R can be identified with the subring S of $\mathbb{C}[t]$ given by

$$S = \{\sum a_i t^i \mid a_1 = 0\} \subset \mathbb{C}[t]$$

using the map $\varphi \colon \mathbb{C}[x,y] \to \mathbb{C}[t]$, $\varphi(f(x,y)) = f(t^2, t^3)$ and the first isomorphism theorem. See HW5Q5a. Prove carefully that $x, y \in R$ are irreducible. Consider the equality $y^2 = x^3$ in R. Alternatively, use Q9.]