Math 611 Homework 5

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All rings are assumed to be commutative with 1 unless explicitly stated otherwise.

(1) Identify the quotient rings with a standard ring.

  (a) $\mathbb{Q}[x]/(x-3)$.

  (b) $\mathbb{R}[x]/(x^2-4x-5)$.

  (c) $\mathbb{R}[x]/(x^2+4)$.

(2) Let R be a ring.

  (a) Show that there is a unique homomorphism $\varphi\colon \mathbb{Z} \to R$ and describe it explicitly. Write $\ker(\varphi) = (n)$, $n \in \mathbb{Z}$, $n \geq 0$. The number n is called the *characteristic* of R.

  (b) Show that if R is an integral domain then $n = 0$ or $n = p$, a prime.

(3) Let $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$, the ring of Gaussian integers. Identify the quotient $\mathbb{Z}[i]/(3+4i)$ with a standard ring.

(4) Suppose R is a ring of characteristic p, a prime. (See Q2 for the definition of the characteristic of a ring.)

  (a) Show that the map $F\colon R \to R$, $\quad F(a) = a^p$ is a ring homomorphism. The homomorphism F is called the *Frobenius homomorphism.*

  (b) Describe F explicitly in the case $R = (\mathbb{Z}/p\mathbb{Z})[x]$, the ring of polynomials in the variable x with coefficients in $\mathbb{Z}/p\mathbb{Z}$.

(5) Identify the kernel and the image of each of the following homomorphisms explicitly.

   (a) $\varphi\colon \mathbb{C}[x,y] \to \mathbb{C}[t]$, $\varphi(f(x,y)) = f(t^2, t^3)$.
   (b) $\psi\colon \mathbb{C}[x,y] \to \mathbb{C}[t]$, $\psi(f(x,y)) = f(t^2-1, t(t^2-1))$.

(6) Let R be a ring. We say an element $a \in R$ is *nilpotent* if $a^n = 0$ for some $n \in \mathbb{N}$.

   (a) Prove that the set $N \subset R$ of nilpotent elements is an ideal of R.
   (b) Show that in the quotient ring R/N there are no nonzero nilpotent elements.

(7) We say a ring R is *local* if there is a unique maximal ideal $M \subset R$.

   (a) Let n be a positive integer, $n \neq 1$. Show that $\mathbb{Z}/n\mathbb{Z}$ is a local ring iff n is a power of a prime.
   (b) Show that if R is local with maximal ideal M then the units $R^\times$ of R are given by $R^\times = R \setminus M$.
   (c) Conversely, show that if R is a ring and $I \subset R$ is an ideal such that every element of $R \setminus I$ is a unit, then R is local with maximal ideal I.

(8) The *formal power series ring* $\mathbb{C}[[x]]$ has elements

$$f = \sum_{i=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \cdots$$

where $a_i \in \mathbb{C}$ for each i. (The adjective *formal* indicates that we do not require that the series converges for any nonzero value of x in $\mathbb{C}$. That is, using the terminology of complex analysis, the radius of convergence may be equal to zero.) Addition and multiplication of formal power series are defined in the obvious way, e.g.,

$$(\sum a_i x^i) \cdot (\sum b_i x^i) = \sum_k (\sum_{i+j=k} a_i b_j) x^k = a_0 b_0 + (a_0 b_1 + a_1 b_0)x + \cdots .$$

(Notice that the coefficients of the product are finite sums, so the product is well-defined.)

(a) Show that $\mathbb{C}[[x]]$ is a local ring with maximal ideal $M = (x)$.

(b) Show that the fraction field of $\mathbb{C}[[x]]$ can be identified with the ring $\mathbb{C}((x))$ of *formal Laurent series*: formal expressions

$$f = \sum_{i=n}^{\infty} a_i x^i$$

for some $n \in \mathbb{Z}$ and $a_i \in \mathbb{C}$.

(9) Let $R = \mathbb{Z}[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in \mathbb{Z}\} \subset \mathbb{R}$, a subring of $\mathbb{R}$. Identify the fraction field $F = \mathrm{ff}(R) \subset \mathbb{R}$ of R explicitly. More precisely, since $\mathbb{Z} \subset R$ we have $\mathbb{Q} = \mathrm{ff}(\mathbb{Z}) \subset F = \mathrm{ff}(R)$. Now the ring operations on F make F into a $\mathbb{Q}$-vector space. Find a basis for F as a $\mathbb{Q}$-vector space.

(10) (Optional) Let R be a noncommutative ring with 1. We say $I \subset R$ is a *two-sided ideal* if $I \subset R$ is an additive subgroup and

$$a \in R, x \in I \Rightarrow ax \in I \text{ and } xa \in I.$$

Then we can define the quotient ring R/I as in the commutative case. However, in the noncommutative case, there are typically very few two-sided ideals.

(a) Let $R = \mathbb{C}\langle x, y\rangle$ be the polynomial ring in noncommuting variables x and y. (Thus an element f of R is a finite sum $\sum a_i w_i$ where $a_i \in \mathbb{C}$ and w_i is a word in x and y, e.g. $xyyxxxy = xy^2x^3y$.) Define a surjective homomorphism $\varphi : R \to \mathbb{C}[x, y]$ from R to the usual commutative polynomial ring. (Then $I = \ker(\varphi) \subset R$ is a two-sided ideal and $R/I \simeq \mathbb{C}[x, y]$.)

(b) Let $R = \mathbb{C}^{n \times n}$ be the ring of $n \times n$ complex matrices. Show that the only two-sided ideals of R are $\{0\}$ and R.

(c) Let R be the *Weyl algebra* given by the polynomial ring $\mathbb{C}\langle x, y\rangle$ in noncommuting variables x and y modulo the two sided ideal generated by the relation $yx - xy - 1$. (Thus the elements f of R may be written uniquely as a finite sum $f = \sum a_{ij} x^i y^j$, $a_{ij} \in \mathbb{C}$, with the usual addition and the multiplication determined by $yx = xy + 1$ together with the associative and distributive laws.) Show that the only two-sided ideals of R are $\{0\}$ and R.

(11) (Optional) Let R be a ring which is commutative but may not have a multiplicative identity. Suppose R is a finite set and there exists an element $a \in R$ which is not a zero divisor. Prove that R has a multiplicative identity.

Hints:

(1) Describe an isomorphism with a product of standard rings using the first isomorphism theorem and/or the Chinese remainder theorem.

(2) Recall that part of our definition of a ring homomorphism φ is that $\varphi(1) = 1$.

(3) Show that the map $\mathbb{Z} \to \mathbb{Z}[i]/(3+4i)$ (from Q2) is surjective and identify the kernel.

(4) (a) Use the binomial theorem. What is the binomial coefficient $\binom{p}{i}$ modulo p? (b) What is Fermat's little theorem?

(5) In each case the kernel is a principal ideal. Find an element of the kernel, and prove it generates the kernel by using $\mathbb{C}[x,y] = (\mathbb{C}[x])[y] = (\mathbb{C}[y])[x]$ and the following division algorithm: Let R be a ring and $a, b \in R[x]$. If b is a monic polynomial (that is, the leading coefficient of b equals 1) then there exist unique $q, r \in R[x]$ such that $a = qb + r$ where $r = 0$ or $\deg(r) < \deg(b)$.

(8) (a) The elements of the set $\mathbb{C}[[x]] \setminus (x)$ are the formal power series f with nonzero constant term. Explicitly construct the inverse of such a power series. Now use Q7(b).

(10) (c) Given $f \in R$ consider the commutators $[x, f] = xf - fx$ and $[y, f] = yf - fy$.

(11) Consider the map $f \colon R \to R$ given by $f(x) = ax$. This is a permutation of the set R (why?)