# Math 611 Homework 4

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(1) Let $G$ be a group such that $|G| = mn$ where $\gcd(m, n) = 1$. Suppose there exists a normal subgroup $H \lhd G$ of order $|H| = m$ and a subgroup $K \leq G$ of order $|K| = n$. Show that $G$ is isomorphic to a semi-direct product of $H$ and $K$.

(2) Express the following groups as a semi-direct product of two non-trivial groups.

  (a) The group $\mathrm{O}(2)$ of orthogonal $2 \times 2$ matrices.

  (b) The symmetric group $S_n$ on $n$ objects, for $n \geq 3$.

  (c) The general linear group $\mathrm{GL}_n(F)$ of invertible $n \times n$ matrices over a field $F$, for $n \geq 2$ and $F \neq \mathbb{Z}/2\mathbb{Z}$.

(3) Show that the general linear group $\mathrm{GL}_n(F)$ of invertible $n \times n$ matrices over a field $F$ is a direct product of two non-trivial groups in the following cases.

  (a) $F = \mathbb{R}$ and $n$ is odd.

  (b) $F = \mathbb{Z}/p\mathbb{Z}$ and $\gcd(n, p - 1) = 1$.

(4) In class we discussed the automorphism group $\mathrm{Aut}(Q_8)$ of the quaternion group $Q_8$.

  (a) Show carefully that $\mathrm{Aut}(Q_8)$ is isomorphic to $S_4$.

  (b) Using your answer to part (a) or otherwise, express $S_4$ as a semi-direct product $(\mathbb{Z}/2\mathbb{Z})^2 \rtimes_\varphi S_3$. What is the homomorphism $\varphi\colon S_3 \to \mathrm{Aut}((\mathbb{Z}/2\mathbb{Z})^2)$?

(5) Compute the number of Sylow $p$-subgroups of $G$ in each of the following cases.

   (a) $p = 2$ and $G = D_{60}$, the dihedral group of symmetries of a regular 60-gon.
   (b) $p = 3$ and $G = S_6$, the symmetric group on 6 objects.
   (c) $p = 5$ and $G = \mathrm{GL}_3(\mathbb{Z}/5\mathbb{Z})$, the general linear group of invertible $3 \times 3$ matrices over $\mathbb{Z}/5\mathbb{Z}$.

(6) What are the possibilities for the number of elements of order 5 in a group $G$ of order 50? Include examples showing that each case occurs.

(7) Classify groups $G$ of order 45.

(8) Let $G$ be a non-abelian group of order 57. Describe $G$ (a) as a semi-direct product and (b) in terms of generators and relations.

(9) Let $G$ be a group of order $|G| = p^a q^b$ where $p$ and $q$ are distinct primes and $a, b \in \mathbb{N}$. Suppose that the order of $p$ in the multiplicative group $(\mathbb{Z}/q\mathbb{Z})^\times$ is greater than $a$. Show that $G$ is isomorphic to the semi-direct product of two non-trivial groups.

(10) Let $G$ be a group of order $|G| = pqr$ where $p, q, r$ are distinct primes. Show that one of the Sylow subgroups of $G$ is normal.

(11) Classify groups $G$ of order (a) $|G| = 18$, (b) $|G| = 28$. (Express the groups as semi-direct products. You should also write the groups in terms of generators and relations and identify them with direct products of known groups where possible.)

(12) Let $G$ be a finite group and let $\varphi \colon G \to S_G$ be the homomorphism given by the action of $G$ on itself by left multiplication. (Here $S_G$ denotes the symmetric group of permutations of the set $G$.)

   (a) Show that $\varphi(g)$ is an odd permutation iff the order $\operatorname{ord}(g)$ is even and $|G|/\operatorname{ord}(g)$ is odd.
   (b) Suppose $|G| = 2m$ where $m$ is odd. Prove that $G$ contains a normal subgroup of index 2.

(13) Let $G = \mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})$ and let $H \leq G$ be a subgroup of order a power of $p$. Prove that there exists $g \in G$ such that $ghg^{-1}$ is upper triangular for all $h \in H$.

Hints:

(1) By a result proved in class, it suffices to show that $H \cap K = \{e\}$ and $HK = G$.

(2) (a) Compare HW3Q3. (b) Consider $A_n \lhd S_n$. (c) Consider $\mathrm{SL}_n(F) \lhd \mathrm{GL}_n(F)$.

(3) Compute the intersection $\mathrm{SL}_n(F) \cap Z(\mathrm{GL}_n(F))$. (b) Recall that the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ is cyclic.

(4) (a) Recall that $S_4$ is isomorphic to the group of rotations of the cube. Consider the cube with vertices $(\pm 1, \pm 1, \pm 1)$ in $\mathbb{R}^3$, so that the centers of the faces are $\pm i$, $\pm j$, $\pm k$, where $i = (1,0,0)$, $j = (0,1,0)$, $k = (0,0,1)$. Note that the quaternion multiplication agrees with the cross product (i.e., $ij = i \times j$ etc.). (b) Recall that $S_3 \simeq \mathrm{GL}_2(\mathbb{Z}/2\mathbb{Z}) = \mathrm{Aut}((\mathbb{Z}/2\mathbb{Z})^2)$.

(5) Recall that all Sylow $p$-subgroups are conjugate. Find one Sylow subgroup and compute the number of conjugate subgroups.

(6) As a special case of Sylow theorem 2, any element of order $p$ is contained in a Sylow $p$-subgroup. What is the classification of groups of order $p^2$?

(12) (a) What is the cycle type of the permutation $\varphi(g)$?

(13) What is a Sylow $p$-subgroup of $\mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})$?