Math 611 Homework 3

Paul Hacking

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(1) Enumerate all the normal subgroups of the symmetric group S_4.

(2) Let G be the group of rotational symmetries of the cube. Recall that there is an isomorphism

$$\varphi \colon G \xrightarrow{\sim} S_4$$

given by considering the action of G on pairs of opposite vertices of the cube. In class we computed the normalizer of $H = \langle (123) \rangle$ in S_4. Give a geometric description of the normalizer in terms of the isomorphism φ.

(3) Let G be the group of isometries of the Euclidean plane $\mathbb{R}^2$ with group law given by composition of functions. (Recall that an *isometry* of $\mathbb{R}^2$ is a function $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ which preserves distances.) Let H be the subgroup of G consisting of all rotations about the origin.

(a) Determine the normalizer of H in G.

(b) Give an explicit description of the homomorphism

$$N(H) \to \mathrm{Aut}(H), \quad g \mapsto (h \mapsto ghg^{-1}).$$

(4) Let G be a group with $|G| = p^n$ for some prime p and $n \in \mathbb{N}$. In class we used the class equation to prove that the center of G is non-trivial, $Z(G) \neq \{e\}$. As a corollary we showed that any group of order p^2 is abelian. In this question we will study a non-abelian group of order p^3: Let G be the subgroup of $\mathrm{GL}_3(\mathbb{Z}/p\mathbb{Z})$ consisting of upper triangular matrices with all diagonal entries equal to 1.

(a) Determine the center $Z(G)$ of G.

(b) Construct an isomorphism from $G/Z(G)$ to a standard group.

(5) Let $n \in \mathbb{N}$.

(a) Show that the map

$$\theta\colon \mathrm{GL}_n(\mathbb{R}) \to \mathrm{GL}_n(\mathbb{R}), \quad \theta(A) = (A^{-1})^T$$

is an automorphism of $\mathrm{GL}_n(\mathbb{R})$. (We use the notation B^T for the transpose of a square matrix B.)

(b) Prove that the automorphism θ is *not* given by conjugation by some element $B \in \mathrm{GL}_n(\mathbb{R})$. That is, there does *not* exist $B \in GL_n(\mathbb{R})$ such that $BAB^{-1} = (A^{-1})^T$ for all $A \in \mathrm{GL}_n(\mathbb{R})$.

(6) Let p be a prime and consider the cyclic subgroup $H = \langle (123 \cdots p) \rangle$ of the symmetric group S_p.

(a) Determine the number of conjugate subgroups of H. Deduce the order of the normalizer $N(H)$ of H in S_p.

(b) Show that the homomorphism

$$\varphi\colon N(H) \to \mathrm{Aut}(H), \quad g \mapsto ghg^{-1}$$

is surjective with kernel H.

(c) Show that $N(H)$ can be generated by two elements, and describe a set of two generators explicitly for $p = 5$.

(7) Let G be a group and $Z(G)$ the center of G. Prove that if $G/Z(G)$ is cyclic then G is abelian (so that $G = Z(G)$).

(8) Let G be a finite group and H a proper subgroup of G.

(a) Show that the union of the conjugate subgroups of H is not equal to G.

(b) Deduce that there is a conjugacy class which is disjoint from H.

(9) Let G be a finite group. Let p be the smallest prime dividing $|G|$. Suppose H is a normal subgroup of G of order p. Show that H is contained in the center of G.

(10) Let G be a group such that $|G| = p^n$ for some prime p and $n \in \mathbb{N}$. Suppose H is a proper subgroup of G. Prove that H is a proper subgroup of its normalizer $N(H)$ in G.

(11) (Optional) Let F be a field and $n \in \mathbb{N}$. In class we introduced the subgroup B of $GL_n(F)$ consisting of upper triangular matrices, i.e., matrices b such that $b_{ij} = 0$ for $i > j$. We observed that B can be realized as a stabilizer subgroup as follows. Let X denote the set of *flags*, that is, n-tuples $(V_1, \ldots, V_n)$ where $V_i \subset F^n$ is a subspace of dimension i for each i, and

$$V_1 \subset V_2 \subset \cdots \subset V_n = F^n.$$

Then $G = GL_n(F)$ acts on X via $g \cdot (V_1, \ldots, V_n) = (g(V_1), \ldots, g(V_n))$. (Note: Since g is invertible, if V is a subspace of F^n then V and $g(V)$ have the same dimension.) And B is the stabilizer in G of the *standard flag*

$$\langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_n \rangle = F^n.$$

(a) Show that G acts transitively on the set X of flags. Deduce that we have a bijection

$$G/B \to X, \quad gB \mapsto g \cdot x_{\text{std}}.$$

where G/B denotes the set of left cosets of B in G (note B is not normal in G so this is not a group) and $x_{\text{std}} \in X$ denotes the standard flag.

(b) Using part (a) or otherwise, determine a formula for the number $|X|$ of flags in case F is a finite field of order q.

(12) (Optional) Let G be a group. The action of G on itself by left multiplication gives an injective homomorphism

$$\varphi\colon G \to S_G, \quad g \mapsto (x \mapsto gx).$$

Here S_G denotes the symmetric group of permutations of the set G (i.e. the set of bijections from G to itself with the group operation given by composition of functions). In particular, if $|G| = n$ then choosing an ordering of the elements of G gives an isomorphism $S_G \simeq S_n$. (This proves Cayley's theorem: every group of order n is isomorphic to a subgroup of S_n.)

In class we considered the following construction: if H is a subgroup of a group G, define the *normalizer* $N(H)$ of H in G by

$$N(H) = \{g \in G \mid gHg^{-1} = H\}.$$

Then $N(H)$ is a subgroup of G and H is a normal subgroup of $N(H)$. In particular, the group $N(H)$ acts on H by conjugation. This action determines a group homomorphism

$$N(H) \to \mathrm{Aut}(H), \quad g \mapsto (h \mapsto ghg^{-1}).$$

In this question we will combine these two constructions to show that any automorphism of a group G is realized by an instance of the second construction: Let G be group and $\varphi\colon G \to S_G$ the injective homomorphism defined above. Let $H = \varphi(G) \leq S_G$ denote the image of φ. Then φ defines an isomorphism from G to H. Consider the normalizer $N(H)$ of H in S_G. Show that if $\theta\colon G \to G$ is an automorphism of G then $\theta \in N(H)$, and the automorphism

$$\psi\colon H \to H, \quad h \mapsto \theta \circ h \circ \theta^{-1}$$

corresponds to the automorphism $\theta\colon G \to G$ under the isomorphism $\varphi\colon G \to H$. That is, we have $\psi = \varphi \circ \theta \circ \varphi^{-1}$.

Hints:

(1) This can be done quickly using the class equation.

(2) This is similar to the case of $H = \langle(1234)\rangle \leq S_4$ we discussed in class (Although it is a little harder to visualize. For instance, what is the intersection of the cube with vertices $(\pm 1, \pm 1, \pm 1)$ with the plane $x + y + z = 1$?).

(5) (b) Recall that $\mathrm{trace}(BAB^{-1}) = \mathrm{trace}(A)$.

(8) Find an upper bound for the number of elements in the union of conjugate subgroups.

(9) Consider the homomorphism $G \to \mathrm{Aut}(H)$, $g \mapsto (h \mapsto ghg^{-1})$.

(10) Use $Z(G) \neq \{e\}$ and induction on n.