Math 611 Homework 2

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September 28, 2015

(1) Let G be a group and $H \leq G$ a subgroup of G of index 2. Prove that H is normal. Give an example to show that a subgroup of index 3 need not be normal.

(2) (a) Describe the partition of the symmetric group S_4 into conjugacy classes.

(b) In class we explained (briefly) that the group G of rotational symmetries of the cube is isomorphic to S_4. (The isomorphism is obtained by considering the action of G on the set of pairs of opposite vertices of the cube.) Determine the geometric interpretation of the conjugacy classes in S_4 under this isomorphism.

(3) (a) Determine the conjugacy classes in the alternating group A_4. Check your answer using the fact that the order of a conjugacy class divides the order of the group.

(b) Show that A_4 does not have a subgroup of order 6.

(4) Let G be a group and $a \in G$ an element. Determine the centralizer $Z(a)$ of a in G and the size of the conjugacy class $C(a)$ of a in G in the following cases.

(a) $(123) \in S_5$.

(b) $(123)(456) \in S_7$.

(c) $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}/5\mathbb{Z})$.

(d) $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$.

(5) A group G of order 21 contains a conjugacy class $C(x)$ of size 3. What is the order of x in G?

(6) Determine the conjugacy classes in the group

$$G = \langle a, b \mid a^3 = b^4 = e, \quad ba = a^{-1}b \rangle.$$

Here you may assume without proof that $|G| = 12$ so that the elements of G can be expressed uniquely as $a^i b^j$ for $0 \le i < 3$ and $0 \le j < 4$.

[Note: This group was studied in HW1Q9.]

(7) (a) Consider the action of $\mathrm{PGL}_2(\mathbb{Z}/3\mathbb{Z})$ on $\mathbb{P}^1_{\mathbb{Z}/3\mathbb{Z}} = \mathbb{Z}/3\mathbb{Z} \cup \{\infty\}$ by Mobius transformations (See Q11(a) for more details.) Use this action to prove that $\mathrm{PGL}_2(\mathbb{Z}/3\mathbb{Z})$ is isomorphic to the symmetric group S_4.

(b) Show that the subgroup $\mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z})$ of $\mathrm{PGL}_2(\mathbb{Z}/3\mathbb{Z})$ is isomorphic to the alternating group A_4.

(8) Classify finite groups G with at most 3 conjugacy classes.

(9) For each of the following statements, give a proof or a counterexample.

(a) If $H \lhd G$ and $K \lhd H$ then $K \lhd G$.

(b) If $H \lhd G$ and $K \le G$ then $H \cap K \lhd K$.

(10) Let G be a finite group and $H \lhd G$ a normal subgroup. Let $a \in H$ be an element. Let $C_H(a)$ denote the conjugacy class of a in H and $C_G(a)$ the conjugacy class of a in G. Let $Z_H(a)$ denote the centralizer of a in H and $Z_G(a)$ the centralizer of a in G. (Then $Z_H(a) = Z_G(a) \cap H$.) Let $q \colon G \to G/H$ be the quotient homomorphism.

(a) Show that $gC_H(a)g^{-1} = C_H(gag^{-1})$ for all $g \in G$.

(b) Show that $C_G(a)$ is a union of $[G/H : q(Z_G(a))]$ distinct conjugacy classes in H of equal size (the orbit of the conjugacy class $C_H(a)$ under conjugation by elements of G).

(c) Now suppose $G = S_n$, the symmetric group on n objects ($n \ge 2$), and $H = A_n$, the alternating group. Deduce that if $Z_{S_n}(a)$ is not contained in A_n then $C_{S_n}(a) = C_{A_n}(a)$, while if $Z_{S_n}(a)$ is contained in A_n then $C_{S_n}(a) = C_{A_n}(a) \cup C_{A_n}((12)a(12))$ is a union of two distinct conjugacy classes in A_n. Give examples showing that both cases occur.

(11) (This question is optional and will not be graded). Let F be a field and n a non-negative integer. Define an equivalence relation $\sim$ on $F^{n+1}\setminus\{0\}$ by

$$(a_0, \dots, a_n) \sim (b_0, \dots, b_n) \iff$$
$$(a_0, \dots, a_n) = \lambda(b_0, \dots, b_n) \text{ for some } 0 \neq \lambda \in F.$$

The *projective space* $\mathbb{P}^n_F$ over F is the set of equivalence classes for this equivalence relation, $\mathbb{P}^n_F = (F^{n+1} \setminus \{0\})/\sim$. (Equivalently, $\mathbb{P}^n_F$ is the set of 1-dimensional subspaces of the vector space F^{n+1}.) The action of $\mathrm{GL}_{n+1}(F)$ on F^{n+1} by left multiplication, $(A, \mathbf{x}) \to A\mathbf{x}$, induces a faithful action of $\mathrm{PGL}_{n+1}(F)$ on $\mathbb{P}^n_F$.

(a) Show that if $n = 1$ then we have a bijection $\mathbb{P}^1_F \to F \cup \{\infty\}$ given by

$$(a_0, a_1) \mapsto \frac{a_1}{a_0}.$$

Show that under this bijection the action of $\mathrm{PGL}_2(F)$ on $\mathbb{P}^1_F$ corresponds to its action on $F \cup \{\infty\}$ by Möbius transformations: for

$$A = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \in \mathrm{PGL}_2(F)$$

and $z \in F \cup \{\infty\}$,

$$A \cdot z = \frac{az + b}{cz + d}.$$

[Note: Usually the term Möbius transformation is used only in the case $F = \mathbb{C}$; we are using it more generally here.]

(b) Show more generally that there is a bijection

$$\mathbb{P}^n_F \to F^n \cup \mathbb{P}^{n-1}_F$$

given by

$$[(a_0, \dots, a_n)] \mapsto \begin{cases} (\frac{a_1}{a_0}, \dots, \frac{a_n}{a_0}) \in F^n & \text{if } a_0 \neq 0. \\ [(a_1, \dots, a_n)] \in \mathbb{P}^{n-1}_F & \text{if } a_0 = 0. \end{cases}$$

(c) (For those who have studied point set topology.) Suppose that $F = \mathbb{R}$ or $F = \mathbb{C}$, give F^{n+1} the usual (Euclidean) topology, and give $\mathbb{P}^n_F$ the quotient topology. Prove that $\mathbb{P}^n_F$ is compact. (This is one of the main reasons projective space is important.)

Review of definitions and notation for matrix groups

Let F be a field and $n \in \mathbb{N}$. The *general linear group* $\mathrm{GL}_n(F)$ denotes the group of invertible $n \times n$ matrices with entries in the field F, and group law given by matrix multiplication. The *special linear group* $\mathrm{SL}_n(F) \triangleleft \mathrm{GL}_n(F)$ is the normal subgroup consisting of matrices with determinant 1. The center $Z := Z(\mathrm{GL}_n(F))$ of $\mathrm{GL}_n(F)$ is the normal subgroup consisting of scalar matrices λI where $0 \neq \lambda \in F$ and I denotes the identity matrix (compare HW1Q6a). The *projective general linear group* $\mathrm{PGL}_n(F)$ is the quotient group $\mathrm{GL}_n(F)/Z$. The *projective special linear group* $\mathrm{PSL}_n(F)$ is the quotient group $\mathrm{SL}_n(F)/Z \cap \mathrm{SL}_n(F)$. In particular $\mathrm{PSL}_n(F)$ is a normal subgroup of $\mathrm{PGL}_n(F)$.

Hints:

(1) Identify the left and right cosets of H in G.

(3) (b) By Q1, a subgroup of A_4 of order 6 is normal. Equivalently, it is a union of conjugacy classes of G.

(7) (a) $\mathrm{PGL}_2(\mathbb{Z}/3\mathbb{Z})$ acts faithfully on $\mathbb{P}^1_{\mathbb{Z}/3\mathbb{Z}} = \mathbb{Z}/3\mathbb{Z} \cup \{\infty\}$ (why?). (More generally, $\mathrm{PGL}_{n+1}(F)$ acts faithfully on $\mathbb{P}^n_F$ for any field F and $n \in \mathbb{N}$.) (b) Note first that by its definition $\mathrm{PSL}_n(F)$ is a subgroup of $\mathrm{PGL}_n(F)$. Compute $|\,\mathrm{PSL}_2(\mathbb{Z}/3\mathbb{Z})| = 12$ and prove that S_4 has a unique subgroup of index 2.

(8) Consider the class equation of such a group.

(9) (a) Try to construct a counterexample. (b) Recall that we say a subgroup H of a group G is *normal* (and write $H \lhd G$) if $gHg^{-1} = H$ for all $g \in G$. In fact, to show that H is normal it suffices to check that $gHg^{-1} \subset H$ for all $g \in G$ (why?).

(11) (c) Show that $\mathbb{P}^n_F$ is the image of a sphere (of some dimension depending on n and F) under a continuous map.