# Math 611 Homework 1

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(1) Let $G$ be a group and $a, b \in G$ elements such that $a$ has order 7 and $a^3 b = b a^3$. Show that $ab = ba$.

(2) Let $G$ be a group of order $|G| = 22$. Let $a, b \in G$ be two elements such that $a \neq e$ and $b$ is not a power of $a$. Show that $G$ is generated by $a$ and $b$.

[Recall that for $G$ a group and elements $a_1, a_2, \ldots, a_n \in G$ the *subgroup generated by* $a_1, a_2, \ldots, a_n$, denoted $\langle a_1, a_2, \ldots, a_n \rangle$, is the smallest subgroup containing $a_1, a_2, \ldots, a_n$. Equivalently $\langle a_1, \ldots, a_n \rangle$ consists of the elements

$$a_{i_1}^{\epsilon_{i_1}} \cdots a_{i_m}^{\epsilon_{i_m}}$$

for $m$ a non-negative integer, $i_1, \ldots, i_m \in \{1, \ldots, n\}$, and $\epsilon_1, \ldots, \epsilon_m \in \{\pm 1\}$. And we say $G$ is generated by $a_1, \ldots, a_n$ if $G = \langle a_1, \ldots, a_n \rangle$.]

(3) Let $G$ be a group of order $|G| = 18$, $G'$ a group of order $|G'| = 15$, and $\varphi \colon G \to G'$ a non-trivial homomorphism. (We say a group homomorphism $\varphi$ is *non-trivial* if $\varphi(g) \neq e$ for some $g \in G$.) What is the order of the kernel

$$\ker(\varphi) := \{g \in G \mid \varphi(g) = e\}$$

of $\varphi$?

(4) Recall that the *order* of an element $g$ of a group $G$ is the least $n \in \mathbb{N}$ such that $g^n = e$ (or $\infty$ if no such $n$ exists). Let $G$ be a group and $a, b \in G$. Show that $ab$ and $ba$ have the same order.

(5) Let $G$ be a group. We say a subgroup $H$ of $G$ is *proper* if $H \neq \{e\}$ and $H \neq G$. Which groups have no proper subgroups?

(6) Let $G$ be a group. The *center* $Z(G)$ of $G$ is the subset of $G$ consisting of elements which commute with every element of $G$, i.e.,

$$Z(G) = \{z \in G \mid zg = gz \quad \forall g \in G\}.$$

The center $Z(G)$ is a normal subgroup of $G$. Determine the center of the following groups.

(a) The group $\mathrm{GL}_n(\mathbb{R})$ of $n \times n$ invertible matrices with real entries.

(b) The symmetric group $S_n$ of permutations of $n$ objects.

(c) The dihedral group $D_n$ of symmetries of a regular $n$-gon ($n \geq 3$).

(7) Let $D_n$ denote the dihedral group of symmetries of the regular $n$-gon, $n \geq 3$.

(a) Let $a$ be counterclockwise rotation by $2\pi/n$ about the center of mass of the polygon and $b$ be reflection in an axis of symmetry of the polygon. Show that $ba = a^{-1}b$.

(b) Show that $D_{2n}$ is isomorphic to $D_n \times \mathbb{Z}/2\mathbb{Z}$ if $n$ is odd.

(c) Show that $D_{2n}$ is *not* isomorphic to $D_n \times \mathbb{Z}/2\mathbb{Z}$ if $n$ is even.

(8) Recall that the *quaternion group* $Q_8$ is the group of order 8 defined by

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\},$$

$$i^2 = j^2 = k^2 = -1,$$

$$ij = k,\ jk = i,\ ki = j,$$

$$ji = -k,\ kj = -i,\ ik = -j.$$

Prove that $Q_8$ is *not* isomorphic to $D_4$ (the group of symmetries of a square).

(9) When we specify a group $G$ by generators and relations we mean the following. Let the generators be denoted $a_1, \ldots, a_n$, and write the relations in the form $r_j = e$, $j = 1, \ldots, m$, where $r_j$ is a word in $a_1^{\pm 1}, \ldots, a_n^{\pm 1}$. Then $G$ is the quotient of the free group $F$ on $a_1, \ldots, a_n$ (whose elements are arbitrary words in $a_1^{\pm 1}, \ldots, a_n^{\pm 1}$) by the smallest normal subgroup $N$ containing the relations $r_1, \ldots, r_m$. Informally, it is the group generated by the set of elements $a_1, \ldots, a_n$ such that the

relations which hold in the group between these elements are those which can be deduced from the given relations $r_1 = \cdots = r_m = e$ using the group axioms. An equivalent formulation is the following "universal property" : given a group $G'$ and elements $a_i' \in G'$ satisfying the relations $r_1 = \cdots = r_m = e$ (with $a_i$ replaced by $a_i'$ for each $i$), there is a unique homomorphism $G \to G'$ such that $a_i \mapsto a_i'$ for each $i$.

Let $G$ denote the group defined by generators and relations as follows:

$$G = \langle a, b \mid a^3 = b^4 = e, \quad ba = a^{-1}b \rangle.$$

(Remark: The group $G$ is a *semidirect product* $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/4\mathbb{Z}$ of $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/4\mathbb{Z}$.)

(a) Show that there is a surjective homomorphism $G \to \mathbb{Z}/4\mathbb{Z}$ with kernel isomorphic to $\mathbb{Z}/3\mathbb{Z}$. In particular, $G$ is a group of order 12.

(b) Prove that no two of the groups $D_6$, $A_4$, and $G$ are isomorphic.

(10) Let $G$ be a group. We say two elements $a, b \in G$ are *conjugate* if there is a $g \in G$ such that $b = gag^{-1}$. Consider the two matrices

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Are $A$ and $B$ conjugate in $\mathrm{GL}_2(\mathbb{R})$? Are $A$ and $B$ conjugate in $\mathrm{SL}_2(\mathbb{R})$? (Recall that $\mathrm{SL}_n(\mathbb{R})$ denotes the normal subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting of matrices with determinant 1.)

(11) Let $G$ be a group and $H$ a subgroup of $G$. Recall that the *left cosets* of $H$ in $G$ are the subsets

$$gH := \{gh \mid h \in H\}$$

where $g \in G$. The left cosets give a partition of $G$. (Two left cosets $gH$ and $g'H$ are equal iff $g^{-1}g' \in H$, and if they are not equal then they are disjoint.) The right cosets $Hg$ are defined similarly and give another partition of $G$. Determine the partitions into left and right cosets of $H$ in $G$ in the following cases.

(a) $G = A_4$, the alternating group on 4 objects, and $H = \langle (123) \rangle$, the subgroup generated by the 3-cycle (123).

(b)

$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R}, \quad x > 0 \right\},$$

a subgroup of $\mathrm{GL}_2(\mathbb{R})$, and

$$H = \left\{ \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \;\middle|\; x \in \mathbb{R}, \quad x > 0 \right\}.$$

To describe the cosets here, identify $G$ with the halfplane $x > 0$ of the $xy$-plane and give a geometric description of the cosets (include a picture).

Hints:

6 (a) Test the commutation relation of a given matrix $Z$ with the elementary matrices $E_{kl}$ having entry 1 in position $(k, l)$ and zeroes elsewhere. (b) Note that $zg = gz$ iff $z = gzg^{-1}$. Now consider the cycle decomposition of $z$.

7 (a) Position the center of mass of the polygon at the origin in $\mathbb{R}^2$ so that its symmetries are realized by matrices and compute explicitly. (b) What is the center of $D_{2n}$?

8 Count the number of elements of each order.

9 (a) Find a convenient "normal form" for the elements of $G$. Compare with the case of the dihedral group. Now use the universal property to define the homomorphism and show that the normal form of an element is unique.

10 The condition $b = gag^{-1}$ can be rewritten $bg = ga$. Now compute explicitly