Math 462 Midterm review questions

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(1) Let S^2 be a sphere in $\mathbb{R}^3$ with center the origin. Let $P = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ and $Q = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ be two points on S^2.

(a) Compute the radius R of S^2.

(b) Find the equation of the plane Π such that $S^2 \cap \Pi$ is the great circle (or spherical line) through P and Q.

(c) What is the shortest path on S^2 from P to Q? Compute its length.

(2) Let S^2 be the sphere in $\mathbb{R}^3$ with center the origin and radius 1. Let Π_L be the plane with equation $x + y + z = 0$ and Π_M the plane with equation $x + 3y + 7z = 0$. Let $L = \Pi_L \cap S^2$ and $M = \Pi_M \cap S^2$ be the associated spherical lines.

(a) The intersection $L \cap M$ consists of two antipodal points P and Q. Determine these points.

(b) Compute the angle between L and M.

(3) Consider the sphere S^2 in $\mathbb{R}^3$ of radius $R = 1$ with center the origin. Let T be the spherical triangle on S^2 with vertices

$$P = (1, 0, 0), \quad Q = \frac{1}{\sqrt{2}}(1, 1, 0), \quad R = \frac{1}{\sqrt{3}}(1, 1, 1).$$

(a) Compute the equations of the spherical lines given by the sides of the triangle T.

(b) Compute the angles of T. [You may assume that all the angles are $\leq \pi/2$].

(c) Deduce the area of T.

(4) Let S^2 be the sphere in $\mathbb{R}^3$ of radius $R = 1$ with center the origin. Let Π be the plane

$$\Pi = \{(x, y, z) \in \mathbb{R}^3 \mid 2x + y + 2z = 1\} \subset \mathbb{R}^3$$

in $\mathbb{R}^3$.

(a) Explain why the intersection $C = S^2 \cap \Pi$ is a spherical circle and identify its spherical center. [See HW1Q5 for the definition of a spherical circle.]

(b) Compute the Euclidean radius and spherical radius of C.

[Note: For a spherical circle there are two antipodal points on the sphere which can be regarded as its center. Here we assume that the center of the spherical circle is chosen so that the spherical radius is $\leq \pi$.]

(5) Let S^2 be the sphere in $\mathbb{R}^3$ with center the origin and radius 1. Let ABC be a spherical triangle on S^2 with side lengths α, β, γ and angles a, b, c.

(a) State the spherical cosine rule.

(b) Use part (a) to prove that $\alpha < \beta + \gamma$.

(6) Let Π_1 and Π_2 be planes in $\mathbb{R}^3$ containing the origin. Let $T_1 \colon \mathbb{R}^3 \to \mathbb{R}^3$ and $T_2 \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the isometries given by reflection in Π_1 and Π_2 respectively.

(a) Explain why the composite isometry $T_2 \circ T_1$ is a rotation. What is the axis of rotation and angle of rotation? [Hint: Reduce to the two dimensional case discussed in HW3Q6.]

(b) Describe the composite explicitly in the case that Π_1 has equation

$$x + 2y + z = 0$$

and Π_2 has equation

$$2x + 3y + z = 0.$$

[Give the axis and angle of rotation. The direction (clockwise or counterclockwise) of rotation may be omitted.]

(7) Give an algebraic formula $T(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for the following isometries of $\mathbb{R}^2$, where A is a 2×2 orthogonal matrix and $\mathbf{b} \in \mathbb{R}^2$ is a vector.

(a) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is rotation about the point $(2, 3)$ through angle $\pi/2$ counterclockwise.

(b) $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is reflection in the line $y = 3 - x$ followed by translation through distance $2\sqrt{2}$ parallel to the line in the direction of increasing x. [This is a glide reflection.]

(8) Give a precise geometric description of the following isometries.

(a)

$$T\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad T\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2}\begin{pmatrix} x + \sqrt{3}y \\ \sqrt{3}x - y \end{pmatrix}.$$

(b)

$$U\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad U\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} x + y \\ y - x \end{pmatrix}.$$

(c)

$$V\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad V\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 - y \\ 7 - x \end{pmatrix}.$$

(d)

$$W\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad W\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3x - 4y + 2 \\ 4x + 3y + 6 \end{pmatrix}.$$

(9) Give an algebraic formula $T(\mathbf{x}) = A\mathbf{x}$ for the following isometries of $\mathbb{R}^3$ (fixing the origin), where A is a 3×3 orthogonal matrix.

(a) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is rotation about the y-axis through angle θ counterclockwise as viewed from $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

(b) $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ is reflection in the plane Π with equation

$$x + 2y + z = 0.$$

(10) Give a precise geometric description of the following isometries of $\mathbb{R}^3$. [In the case of a rotation or rotary reflection, the direction (counter-clockwise/clockwise) of rotation may be omitted.]

(a)

$$T\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ -x \\ y \end{pmatrix}$$

(b)

$$U\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad U\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -x \\ z \\ y \end{pmatrix}$$

(c)

$$V\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad V\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ y \\ -x \end{pmatrix}$$

(d)

$$W\colon \mathbb{R}^3 \to \mathbb{R}^3, \quad W\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ -1 & 2 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(11) Let $T_1\colon \mathbb{R}^3 \to \mathbb{R}^3$ and $T_2\colon \mathbb{R}^3 \to \mathbb{R}^3$ be two isometries of $\mathbb{R}^3$ fixing the origin. Suppose T_1 and T_2 are rotations. Show carefully that the composite $T_2 \circ T_1$ is another rotation. [Hint: Consider determinants.]

(12) Give a geometric description of the isometries of the sphere S^2 and identify the fixed locus in each case. [The *fixed locus* of an isometry $T\colon S^2 \to S^2$ of the sphere is the set of points $\mathbf{x} \in S^2$ such that $T(\mathbf{x}) = \mathbf{x}$.]