Tuesday 4/28/15

MATH 462 Final Review Solutions.

Q1 (a)

In general, if $z_1, z_2, z_3 \in \mathbb{C}$ are distinct complex numbers, then

$$f(z) = \frac{z-z_1}{z-z_2} \Big/ \frac{z_3-z_1}{z_3-z_2}$$

is the Mobius transformation such that

$$f(z_1) = 0$$
$$f(z_2) = \infty$$
$$f(z_3) = 1.$$

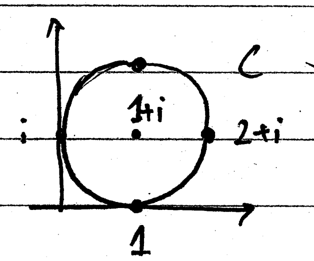
Applying this in our case gives

$$f(z) = \frac{z-2}{z-(1+i)} \Big/ \frac{4-\cancel{(1+i)}2}{4-(1+i)}$$

$$= \frac{z-2}{z-(1+i)} \cdot \frac{3-i}{2}$$

$$= \frac{(3-i)z + (-6+2i)}{2z + (-2-2i)}$$

(b)



circle, center $1+i$, radius 1.

Mobius transformations send circles & lines to circles and lines. And a circle or line is uniquely determined by 3 points lying on it.

So, to find a MT g sending C to $\mathbb{R} \cup \{\infty\}$, we can pick 3 points on C & write down the MT sending those 3 points to 3 points on $\mathbb{R} \cup \{\infty\}$, for example $0, 1,$ and ∞.

For example : $g(i) = 0$

$i, 1, 2+i \in C$ $\quad g(1) = \infty$

(see picture) $\quad g(2+i) = 1$

$$\Rightarrow \quad g(z) = \frac{z-i}{z-1} \Big/ \frac{(2+i)-i}{(2+i)-1}$$

$$= \frac{z-i}{z-1} \cdot \frac{1+i}{2}$$

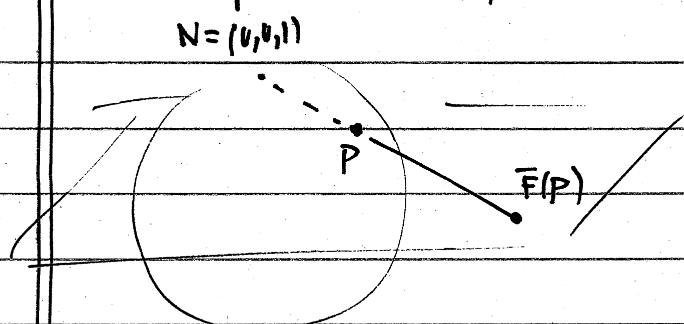
$$= \frac{(1+i)z + (1-i)}{2z-2} \qquad \dagger$$

(Note: g is not uniquely determined by the property that $g(C) = \mathbb{R} \cup \{\infty\}$, so $\dagger$ is NOT the only possible answer.)

Q2. (a)

$C = \Pi \cap S^2$ where $\Pi = \{(x,y,z) \mid x+y+2z = 2\}$
and $S^2 = \{(x,y,z) \mid x^2+y^2+z^2 = 1\}$

$F : S^2 \to \mathbb{R}^2 \cup \{\infty\}$ is stereographic projection from the north pole $N = (0,0,1)$ onto the xy-plane.



, & $F(N) = \infty$.

$F(C) = \{Q \in \mathbb{R}^2 \cup \{\infty\} \mid Q = F(P) \text{ for some } P \in C\}$
$= \{Q \in \mathbb{R}^2 \cup \{\infty\} \mid F^{-1}(Q) \in C\} = \{Q \in \mathbb{R}^2 \cup \{\infty\} \mid F^{-1}(Q) \in \Pi\}$

Now use the formula for F^{-1} to describe $F(C)$ ~~explicitly~~ explicitly:

$$F^{-1}(u,v) = \frac{(2u, 2v, u^2+v^2-1)}{u^2+v^2+1}$$

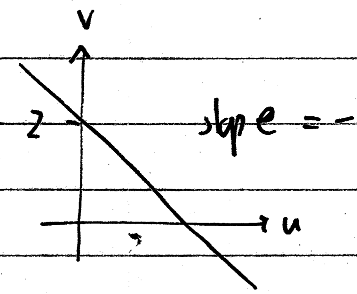
Eqn of Π: $\quad x+y+2z = 2.$

So $F^{-1}(u,v) \in \Pi \iff \dfrac{2u}{u^2+v^2+1} + \dfrac{2v}{u^2+v^2+1} + \dfrac{2\cdot(u^2+v^2-1)}{u^2+v^2+1} = 2$

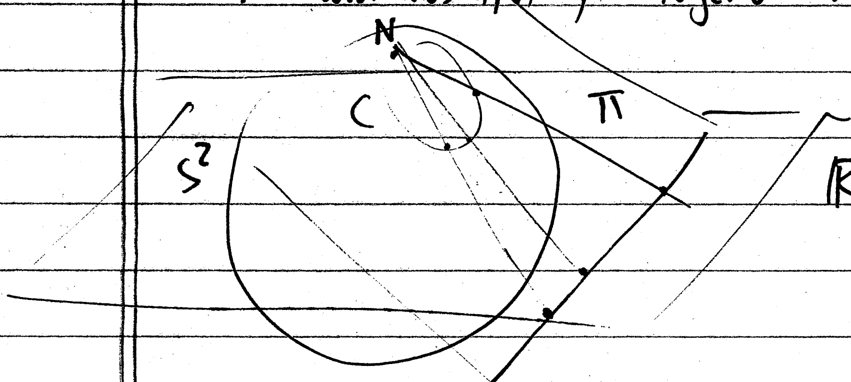
$\iff 2u+2v+2(u^2+v^2-1) = 2(u^2+v^2+1)$

$\iff 2u+2v-2 = 2$

$v = 2-u$



$\therefore$ $F(C)$ is the line $v = 2-u$ in $\mathbb{R}^2$ (with coordinates u,v), together with ∞.



$C = \Pi \cap S^2$ xy-plane

The circle C passes through N.

The image $F(C)$ is the line $\Pi \cap \mathbb{R}^2$, together with $\infty = F(N)$.

(b) $C = \Pi \cap S^2$, $\quad \Pi$ has eq. $4x+2y+5z=6.$

We determine $F(C)$ by same method as in part (a)

$$4\cdot\left(\frac{2u}{u^2+v^2+1}\right)+2\cdot\left(\frac{2v}{u^2+v^2+1}\right)+5\left(\frac{u^2+v^2-1}{u^2+v^2+1}\right)=6.$$

$$8u+4v+5(u^2+v^2-1)=6(u^2+v^2+1)$$
$$8u+4v-5=u^2+v^2+6$$
$$0=u^2+v^2-8u-4v+11$$

"complete the square" $0=(u-4)^2+(v-2)^2-4^2-2^2+11$

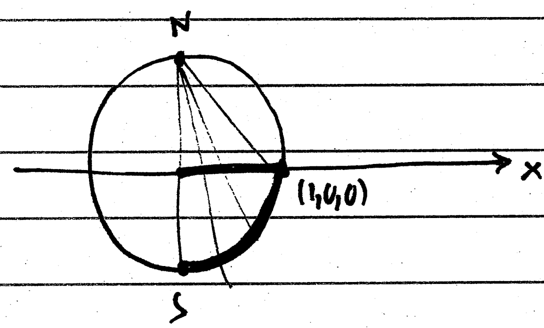
$$(u-4)^2+(v-2)^2=9=3^2$$

— circle, centre $(4,2)$, radius 3. is image $F(C)$ of C under F.

Q3. (a)

Under stereographic projection, the shortest path from $S=(0,0,-1)$ to $(1,0,0)$ maps to the line segment from the origin $(0,0)$ to the point $(1,0)$ in the xy-plane.

- Picture of slice $y=0$:



Now, parameterize this line segment, eg. $\gamma:[0,1]\to\mathbb{R}^2$, $\gamma(t)=(x(t),y(t))=(t,0)$.

Apply give formula: $\text{length}(\gamma)=\displaystyle\int_0^1\frac{2\sqrt{x'(t)^2+y'(t)^2}}{1+x(t)^2+y(t)^2}\,dt$

$$=\int_0^1\frac{2\sqrt{1^2+0^2}}{1+t^2+0^2}\,dt=\int_0^1\frac{2}{1+t^2}\,dt$$

$$=\Big[2\tan^{-1}(t)\Big]_0^1=2\left(\pi/4-0\right)=\pi/2.$$

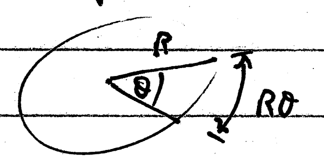
This checks with our earlier calculations in spherical geometry:

The shortest path from $S = (0,0,-1)$ to $(1,0,0)$

is an arc of a great circle corresponding to an angle $\theta = \pi/2$,

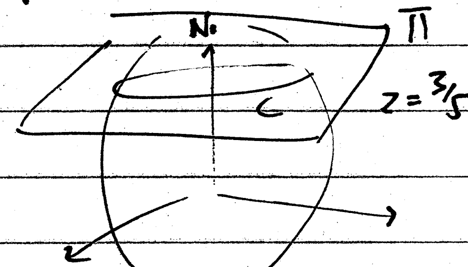
so has length $R \cdot \theta = 1 \cdot \pi/2 = \pi/2$.

R = radius of $S^2 = 1$

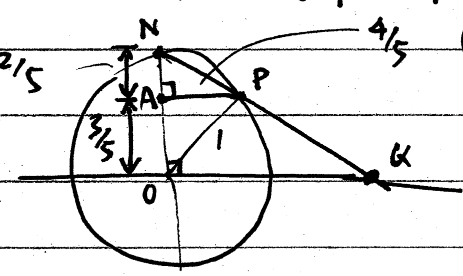


(b) $C = S^2 \cap \Pi$, Π has eq. $z = 3/5$.

$F(C)$ is a circle in $\mathbb{R}^2$ with radius r determined by following picture



(draw slice of stereographic projection):



$$r = OQ = AP \cdot ON / AN = 4/5 \cdot 1 / 2/5 = 2.$$

(or, can compute $F(C)$ as in Q2.)

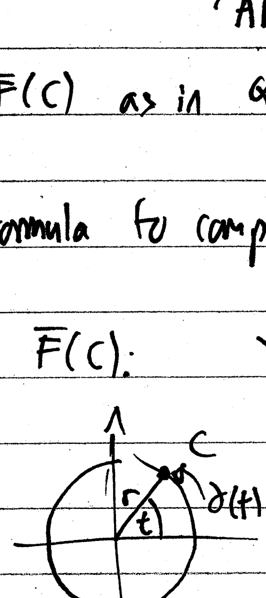
Now use give formula to compute length of $F(C)$.

First parametrize $F(C)$: $\gamma : [0, 2\pi] \longrightarrow \mathbb{R}^2$

$\gamma(t) = (x(t), y(t)) = (r\cos t, r \sin t)$,

$r = 2$.

(see above)



$$\Rightarrow \text{length}(\gamma) = \int_0^{2\pi} \frac{2\sqrt{x'(t)^2+y'(t)^2}}{1+x(t)^2+y(t)^2}\,dt \qquad \begin{aligned} x(t) &= 2\cos t \\ y(t) &= 2\sin t \end{aligned}$$

$$= \int_0^{2\pi} \frac{2\sqrt{(-2\sin t)^2+(2\cos t)^2}}{1+(2\cos t)^2+(2\sin t)^2}\,dt \qquad \begin{aligned} x'(t) &= -2\sin t \\ y'(t) &= 2\cos t \end{aligned}$$

$(\cos t)^2+(\sin t)^2 = 1$

$$= \int_0^{2\pi} 2\cdot\frac{2}{1+4}\,dt = 2\pi\cdot\frac{4}{5} = 8\pi/5.$$

This checks with direct spherical calculation:
C is a Euclidean circle in the plane Π
w/ radius $AP = 4/5$ (see diagram),
so has circumference $2\pi\cdot 4/5 = 8\pi/5$.

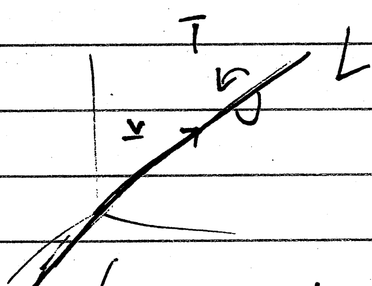
Q4. Recall that a quaternion

$$q = \cos(\theta/2) + \sin(\theta/2)\,\underline{v} \quad \in \mathbb{H}$$

where $\underline{v} \in \mathbb{R}^3 \subset \mathbb{H}$, $\|\underline{v}\| = 1$

determines the rotation $T : \mathbb{R}^3 \to \mathbb{R}^3$

about the axis L in the direction of $\underline{v}$, through angle θ CCW as viewed from $\underline{v}$.



Moreover, if q_1 determines T_1 & q_2 determines T_2,

then q_2q_1 determines $T_2\circ T_1$.

(i.e., composition of rotations corresponds to multiplication of quaternions)

In this example:

$T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ rotation about x axis thru $\pi/2$ ccw.

$\Rightarrow \underline{v}_1 = i = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \theta_1 = \pi/2$

$\Rightarrow q_1 = \cos(\theta_1/2) + \sin(\theta_1/2)\, i = \frac{1}{\sqrt{2}}(1+i)$

$\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}.$

$T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ rotation about z axis thru $\pi/2$ ccw.

$\Rightarrow q_2 = \frac{1}{\sqrt{2}}(1+k)$ similarly.

Now $T_2 \circ T_1$ is determined by

$$q_2 q_1 = \frac{1}{\sqrt{2}}(1+k)\frac{1}{\sqrt{2}}(1+i) = \frac{1}{2}(1+k+i+\underbrace{ki}_{=j})$$

$$= \frac{1}{2}(1+i+j+k)$$

$$= \cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right).\underline{v}$$

$\cos\left(\theta/2\right) = 1/2, \quad 0 \le \theta/2 \le \pi \Rightarrow \theta/2 = \pi/3 \Rightarrow \theta = 2\pi/3$

Now $\sin(\theta/2) = \sin(\pi/3) = \frac{\sqrt{3}}{2}, \quad \underline{v} = \frac{\frac{1}{2}(i+j+k)}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}(i+j+k).$

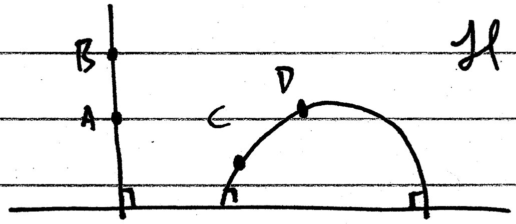
So $T_2 \circ T_1$ is rotation thru angle $2\pi/3$ ccw about the axis L thru the origin in direction $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Q5. $\mathcal{H} = \{x+iy \mid y > 0\} \subset \mathbb{C}$ the upper half plane model of the hyperbolic plane.

If $\gamma: [a,b] \to \mathcal{H}$ is a path, the hyperbolic length of γ is defined by

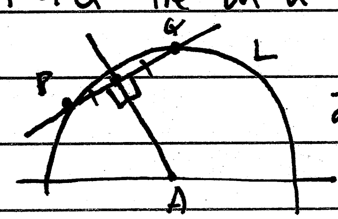
$$\text{length}(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\, dt$$

For this notion of length, the shortest paths between points are given by segments of <u>hyperbolic lines</u> : vertical lines and semicircles with center on the x-axis



shortest paths in $\mathcal{H}$

(a) To find the hyperbolic line L thru two points $P, Q \in \mathcal{H}$, construct the perpendicular bisector of PQ, & find its intersection pt A w/ the x-axis, then L is the semicircle center A & radius $AP = AQ$. (This works unless P & Q lie on a vertical line, then L is this vertical line.)



In our case $P = 1+i$, $Q = 5+3i$
$= (1,1)$ $= (5,3)$

Midpoint M of PQ : $M = \frac{1}{2}((1,1)+(5,3)) = (3,2)$

Direction of PQ : $\underline{v} = (5,3)-(1,1) = (4,2)$

Perpendicular direction: $\underline{w} = (+2,-4)$ $(\underline{v}\cdot\underline{w} = 0)$

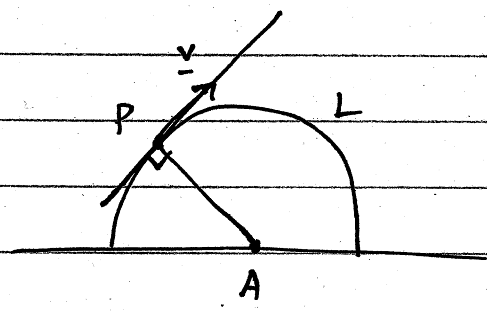
$\therefore$ parametrization of perpendicular bisector : $(3,2) + t(2,-4),\ t \in \mathbb{R}$.
$\parallel$
$(3+2t, 2-4t)$

Intersection point w/ x-axis: $A = (3+2t, 2-4t) = (?, 0)$

$\Rightarrow t = \frac{1}{2}, \quad A = (4, 0)$

$\therefore$ L is semicircle center $A = (4,0)$, radius $AP = \sqrt{(1-4)^2 + (1-0)^2} = \sqrt{10}$. $P = (1,1)$

(b) Similarly, can find hyperbolic line L thru a given point P in direction $\underline{v} \in \mathbb{R}^2$ as follows :–



radius is perpendicular to tangent.

In our case: $P = 2i = (0,2)$

$\underline{v} = (1,3)$

Perpendicular direction $\underline{w} = (3,-1)$ $\quad (\underline{v} \cdot \underline{w} = 0)$

Parametrization of line AP $\quad (0,2) + t \cdot (3,-1) = (3t, 2-t), \ t \in \mathbb{R}$

Solve for A: $\quad A = (3t, 2-t) = (?, 0)$

$\Rightarrow t = 2, \quad A = (6,0)$

$\therefore$ L is semicircle center $A = (6,0)$, radius $AP = \sqrt{(6-0)^2 + (0-2)^2} = \sqrt{40}$.

Q6. (a) Recall that

$(\dagger) \quad f : \mathcal{H} \to \mathcal{H}, \quad f(z) = \dfrac{az+b}{cz+d} \quad a,b,c,d \in \mathbb{R}, \ ad-bc > 0$

is a hyperbolic isometry.

We need to find a hyperbolic isometry such that

$f(2+3i) = 1+4i$

For this, can use isometry of the form $f(z) = az + b$, $a, b \in \mathbb{R}$, $a > 0$. (this is the special case $c = 0, d = 1$ of $\dagger$ above) – a scaling followed by a translation.

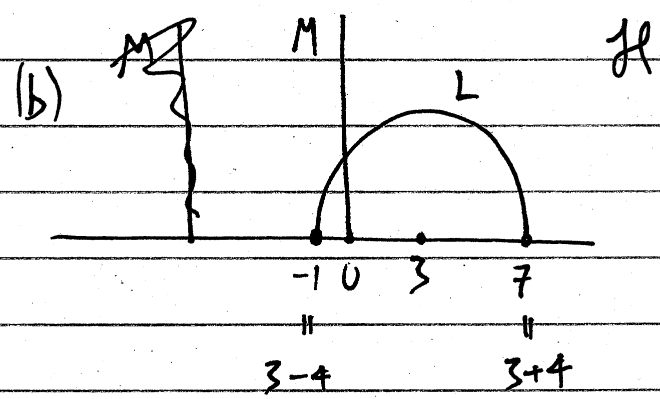
Solve for a & b

$$f(2+3i) = 1+4i$$
$$a(2+3i) + b = 1+4i$$
$$(2a+b) + (3a)i = 1+4i$$

$2a+b = 1,\ 3a = 4 \implies a = 4/3,\quad b = 1-2a = 1 - 8/3 = -5/3$

$$\boxed{f(z) = \tfrac{4}{3}z - \tfrac{5}{3}.}$$

(b)



Want g hyperbolic isometry such that $g(L) = M$.

We can send the endpoints of L to 0 & ∞ using

$$g(z) = -\left(\frac{z-(-1)}{z-7}\right) = \frac{-z-1}{z-7}$$

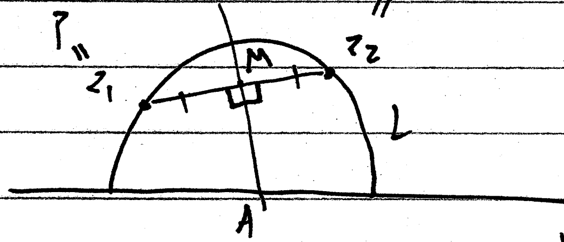
the minus sign is needed here to satisfy the condition $ad-bc>0$ in $(\dagger)$.

Then $g(L)$ is a line (because MT's send circles & lines to circles & lines, and we get a line precisely when one point is sent to ∞; here $g(7) = \infty$ by construction),

$g(L)$ passes through $g(-1) = 0$, and is perpendicular to the x-axis there (because g preserves angles and preserves the x-axis).

So $g(L)$ is the y-axis M as required.

Q7. (a) We use the same method as in Q5(a) to find the hyperbolic line thru $z_1 = -4+3i$ and $z_2 = 3+4i$



$M = \frac{1}{2}(z_1+z_2) = \frac{1}{2}(-1+7i)$

$= \frac{1}{2}(-1,7)$

Direction of line segment $z_1 - z_2$:

$\underline{v} = (3,4) - (-4,3) = (7,1)$

~~Parametrization of line AM:~~ $\cancel{\tfrac{1}{2}(-1,7) + t\cdot(7,1)}$

$\cancel{= (-\tfrac{1}{2}+7t, \tfrac{7}{2}+t}$

Perpendicular direction: $\underline{w} = (-1,7)$

$t \in \mathbb{R}$

Parametrization of line AM: $\frac{1}{2}(-1,7) + t\cdot(-1,7)$

$= (-\frac{1}{2} - t, \frac{7}{2} + 7t) \overset{\text{solve for } A}{=} (?,0) = A$

$\Rightarrow t = -\frac{1}{2}$, $A = (0,0)$.

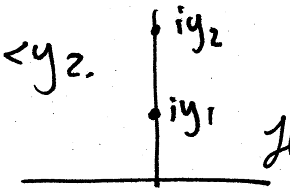
$\Rightarrow L$ is semicircle, center $(0,0)$, radius $AP = \sqrt{(-4)^2+3^2}$
$= 5$.

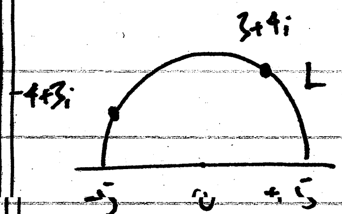
(b) To compute the hyperbolic distance $d_{\mathcal{H}}(z_1,z_2)$,

first write down a hyperbolic isometry $f: \mathcal{H} \to \mathcal{H}$

sending L to the y-axis, then use the known formula

$d_{\mathcal{H}}(iy_1, iy_2) = \ln(y_2/y_1)$ # for $y_1 < y_2$.





We can take

$$f(z) = -\frac{(z-(-5))}{z-5} \quad \text{as in Q6(b),}$$

(then $f(L) = y$-axis)

Now compute

$$f(-4+3i) = \frac{-1-3i}{-9+3i} = \frac{(-1-3i)(-9-3i)}{(-9)^2+3^2}$$

$$= \frac{30i}{90} = \frac{1}{3}i$$

$$f(3+4i) = \frac{-8-4i}{-2+4i} = \frac{(-8-4i)(-2-4i)}{(-2)^2+4^2}$$

$$= \frac{40i}{20} = 2i$$

$$\therefore\ d_{\mathcal{H}}(-4+3i, 3+4i) \underset{f \text{ isometry}}{=} d_{\mathcal{H}}(f(-4+3i), f(3+4i)) = d_{\mathcal{H}}(\tfrac{1}{3}i, 2i)$$

$$\overset{\#}{=} \ln\left(2/\tfrac{1}{3}\right)$$

$$= \boxed{\ln 6}.$$

(c) Parametrize the line segment from $z_1 = -4+3i$ to $z_2 = 3+4i$ as in the Hint:

$$\gamma(t) = -4+3i + t\,(3+4i-(-4+3i)), \quad 0 \le t \le 1$$

$$= -4+3i + t\,(7+i)$$

$$= (-4+7t) + (3+t)i, \text{ i.e., } x(t) = -4+7t,\ y(t) = 3+t$$

Now use the formula to compute the hyperbolic length of the line segment:–

$$\text{length}(\gamma) = \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\,dt$$

$$= \int_0^1 \frac{\sqrt{7^2 + 1^2}}{3+t}\,dt$$

$$= \sqrt{50}\left[\ln|3+t|\right]_0^1 = \sqrt{50}\,(\ln 4 - \ln 3)$$

$$= \sqrt{50}\,\ln(4/3).$$

Finally compute $\ln 6 = 1.79$

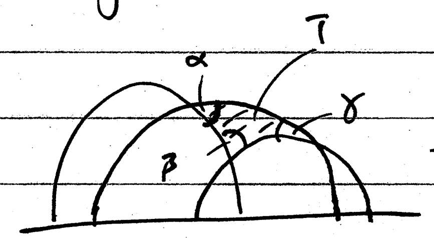
$$\underbrace{\sqrt{50}\,\ln(4/3)}_{\overset{(c)}{=}\ \text{length}(\gamma)} = 2.03 > \underbrace{1.79}_{\overset{(b)}{=}\ d_{\mathcal{H}}(z_1, z_2)}$$

(Note: No calculators are allowed on the exam, so I won't ask you to compute numerical values like this.)

Q8. If $T \subset \mathcal{H}$ is a hyperbolic triangle then

$$\text{Area}(T) = \pi - (\alpha + \beta + \gamma)$$

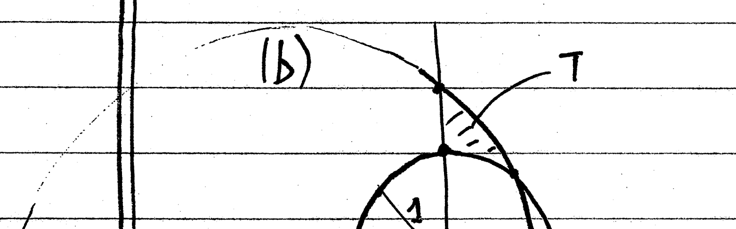
where α, β, γ are the angles of T



~~(a) Area(T) > 0 =>~~

(a) $\alpha, \beta, \gamma > 0 \implies \text{Area}(T) = \pi - (\alpha + \beta + \gamma) < \pi$

(b)



Now we assume two sides of T are given by the y-axis & the semicircle center the origin, & radius 1.

Then observe the angle α between

these two sides is $\alpha = \pi/2$.

So $\text{Area}(T) = \pi - (\alpha+\beta+\gamma) = \pi - (\pi/2 + \beta+\gamma)$

$= \pi/2 - \beta - \gamma < \pi/2$, because $\beta, \gamma > 0$.

Q9. We follow the Hint:

$h(z) = az+b, \quad a, b \in \mathbb{R}, \quad a > 0,$

$h(L) = C$ — L semicircle center $3=(3,0)$, radius 2; C semicircle center $0=(0,0)$, radius 1

$\therefore \quad h(z) = \frac{1}{2}(z-3) = \frac{1}{2}z - \frac{3}{2}$

Now $f(z) = h^{-1}(g(h(z)))$

where $g(z) = \frac{z}{|z|^2} = \frac{1}{\bar{z}}$ is the hyperbolic reflection in C, i.e., inversion in C.

Finally, compute:

$w = h(z) = \frac{1}{2}(z-3) \Rightarrow z = \underline{h^{-1}(w) = 2w+3}$,

so $f(z) = h^{-1}\left(\frac{1}{\cancel{\frac{1}{2}z - \frac{3}{2}}}\right) = 2\left(\frac{1}{\frac{1}{2}\bar{z} - \frac{3}{2}}\right) + 3$

$$= \boxed{\frac{4}{\bar{z}-3} + 3} = \boxed{\frac{3\bar{z}-5}{\bar{z}-3}}$$

Q10. Here we use the geometric classification of the orientation preserving isometries of $\mathcal{H}$ worked out in class:-

isometry	# fixed points in $\mathcal{H}$	# fixed points in $\partial\mathcal{H}$ (= x-axis)
hyperbolic rotation	1	0
horolation	0	1
hyperbolic translation	0	2.

The orientation preserving isometries are given algebraically by $f(z) = \frac{az+b}{cz+d}$ $a,b,c,d \in \mathbb{R}$, $ad-bc > 0$.

Given such a formula, we can determine the geometric type of f by solving the equation

$$f(z) = z$$

and using the table.

(a) $f(z) = \frac{1+z}{1-z}$

Solve $f(z) = z$:

$$\frac{1+z}{1-z} = z$$

$$1+z = z - z^2$$

$$z^2 + 1 = 0$$

$$\Rightarrow z = \pm i$$

$\therefore$ 1 fixed point $i \in \mathcal{H}$ $\Rightarrow$ hyperbolic rotation, center $i = (0,1)$

(b) $f(z) = \frac{z}{2z-1}$

Solve $f(z) = z$:

$$\frac{z}{2z-1} = z$$

$$z = 2z^2 - z$$

$$0 = 2z^2 - 2z$$

$$0 = 2z(z-1)$$

$$z = 0 \text{ OR } 1.$$

$\therefore$ 2 fixed points $0, 1 \in \partial\mathcal{H}$ $\Rightarrow$ hyperbolic translation.

(c) $f(z) = \frac{5z-18}{2z-7}$

Solve $f(z) = z$: $\frac{5z-18}{2z-7} = z$

$$5z-18 = 2z^2-7z$$
$$0 = 2z^2 - 12z + 18$$
$$0 = z^2 - 6z + 9$$
$$0 = (z-3)^2$$
$$\therefore \quad z = 3.$$

1 fixed point $\in \partial\mathcal{H}$ $\Rightarrow$ horolation.

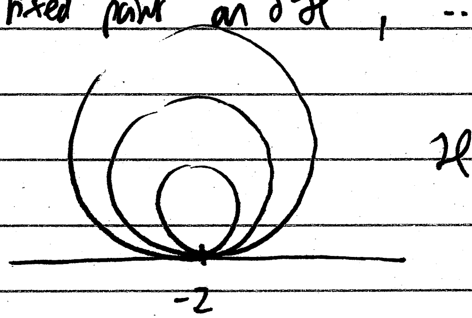
Q11. $T: \mathcal{H} \to \mathcal{H}$, $T(z) = \frac{z-4}{z+5}$

(a) As in Q10, first find fixed points of T to determine type of isometry :-

$T(z) = z$. $\frac{z-4}{z+5} = z$

$$z-4 = z^2+5z$$
$$0 = z^2+4z+4$$
$$0 = (z+2)^2$$
$$z = -2.$$

1 fixed point on $\partial\mathcal{H}$, $\therefore$ T is horolation.



Applying T moves points along the pictured circles (circles tangent to $\partial\mathcal{H}$ = x-axis at the fixed point)

b) Let $f: \mathcal{H} \to \mathcal{H}$ send the fixed point $z=-2$ of the translation T to ∞.

Then $S = fTf^{-1}$ is a Euclidean translation parallel to the x-axis :-

We can take $f(z) = \dfrac{-1}{z+2} = \dfrac{0\cdot z - 1}{1 \cdot z + 2}$

Now compute $S = fTf^{-1}$ $\left(T(z) = \dfrac{z-4}{z+5} \right)$

either directly, or using matrices*

inverse of $\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$, w/ scalar factor $1/\det$ omitted

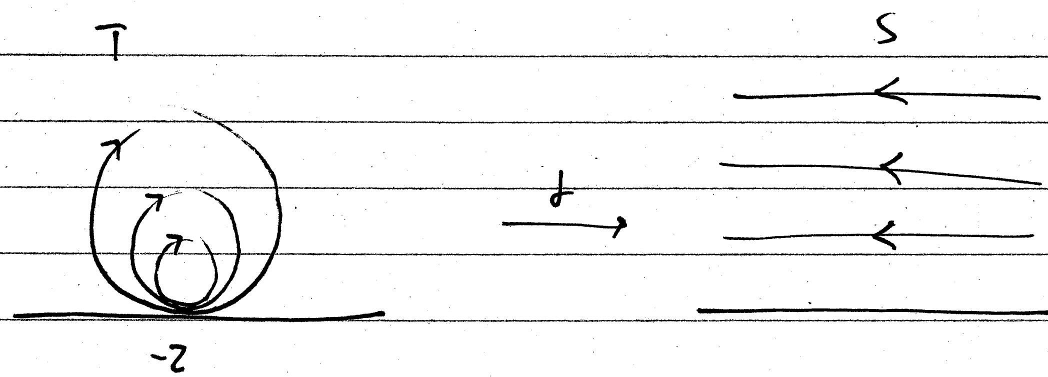
$$\begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & -4 \\ 1 & 5 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & -5 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & -1 \\ 0 & 3 \end{pmatrix}$$

$$\Rightarrow \quad S(z) = (3z-1)/3 = \boxed{z - 1/3.} \quad \text{translation.}$$

Picture



[* recall : Mobius transformations $f(z) = \dfrac{az+b}{cz+d}$ correspond to invertible 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ up to a scalar factor, and composition of MT's corresponds to multiplication of matrices]

Q12. $T: \mathcal{H} \to \mathcal{H}$, $T(z) = \dfrac{z-8}{z-5}$

(a) Find fixed points $T(z) = z$

$$\frac{z-8}{z-5} = z$$

$$z-8 = z^2 - 5z$$

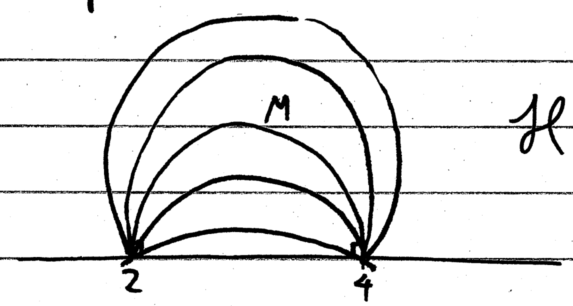
$$0 = z^2 - 6z + 8$$

$$0 = (z-2)(z-4)$$

$$z = 2 \text{ OR } 4.$$

2 fixed points on $\partial\mathcal{H}$ $\Rightarrow$ hyperbolic translation.

Picture



Applying T moves points along pictured circles (circles passing thru fixed points of T)

Note: One of these circles M is a hyperbolic line, (a semicircle with center on the x-axis $= \partial\mathcal{H}$) the others are not.

(b) Let $f: \mathcal{H} \to \mathcal{H}$ be a hyperbolic isometry sending the fixed points of T to 0 & ∞

Then $S = fTf^{-1}$ is a Euclidean scaling $S(z) = az$, $a > 0$.

:— We can take $f(z) = -\dfrac{(z-2)}{(z-4)}$ (as in Q6 b)

$$= \frac{-z+2}{z-4}$$

Now compute S using matrices:

$$\underbrace{\begin{pmatrix} -1 & 2 \\ 1 & -4 \end{pmatrix}\begin{pmatrix} 1 & -8 \\ 1 & -5 \end{pmatrix}}\begin{pmatrix} -1 & 2 \\ 1 & -4 \end{pmatrix}^{-1} = \begin{pmatrix} +1 & -2 \\ -3 & 12 \end{pmatrix}\begin{pmatrix} -4 & -2 \\ -1 & -1 \end{pmatrix} \cdot \frac{1}{\det}$$

can ignore scalar factor

$$= \begin{pmatrix} -2 & 0 \\ 0 & -6 \end{pmatrix}$$

$$S(z) = \frac{-2z}{-6} = \frac{1}{3} z.$$

Picture: